CHAPTER

15

# PROBABILITY

Mathematicians first studied probability by looking at situations involving games of chance. Today, probability is used in a wide variety of fields. In medicine, it helps us to determine the chances of catching an infection or of controlling an epidemic, and the likelihood that a drug will be effective in curing a disease. In industry, probability tells us how long a manufactured product should last or how many defective items may be expected in a production run. In biology, the study of genes inherited from one's parents and grandparents is a direct application of probability. Probability helps us to predict when more tellers are needed at bank windows, when and where traffic jams are likely to occur, and what kind of weather we may expect for the next few days. While the list of applications is almost endless, all of them demand a strong knowledge of higher mathematics.

As you study this chapter, you will learn to solve problems such as the following: A doctor finds that, as winter approaches, 45% of her patients need flu shots, 20% need pneumonia shots, and 5% need both. What is the probability that the next patient that the doctor sees will need either a flu shot or a pneumonia shot?

Like the early mathematicians, we will begin a formal study of probability by looking at games and other rather simple applications.

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## 15-1 EMPIRICAL PROBABILITY

**Probability** is a branch of mathematics in which the chance of an event happening is assigned a numerical value that predicts how likely that event is to occur. Although this prediction tells us little about what may happen in an individual case, it can provide valuable information about what to expect in a large number of cases.

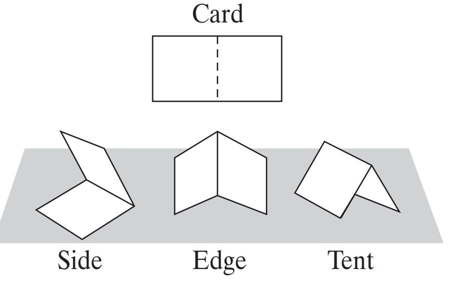
A decision is sometimes reached by the toss of a coin: "Heads, we'll go to the movies; tails, we'll go bowling." When we toss a coin, we don't know whether the coin will land with the head side or the tail side facing upward. However, we believe that heads and tails have equal chances of happening whenever we toss a fair coin. We can describe this situation by saying that the probability of heads is $\frac{1}{2}$ and the probability of tails is $\frac{1}{2}$, symbolized as:

$$P(\text{heads}) = \tfrac{1}{2} \text{ or } P(\text{H}) = \tfrac{1}{2} \qquad\qquad P(\text{tails}) = \tfrac{1}{2} \text{ or } P(\text{T}) = \tfrac{1}{2}$$

Before we define probability, let us consider two more situations.

1. Suppose we toss a coin and it lands heads up. If we were to toss the coin a second time, would the coin land tails up? Is your answer "I don't know"? Good! We cannot say that the coin must now be tails because we cannot predict the next result with certainty.

2. Suppose we take an index card and fold it down the center. If we then toss the card and let it fall, there are only three possible results. The card may land on its side, it may land on its edge, or it may form a tent when it lands. Can we say $P(\text{edge}) = \frac{1}{3}$, $P(\text{side}) = \frac{1}{3}$, and $P(\text{tent}) = \frac{1}{3}$?




Again, your answer should be "I don't know." We cannot assign a number as a probability until we have some evidence to support our claim. In fact, if we were to gather evidence by tossing this card, we would find that the probabilities are *not* $\frac{1}{3}$, $\frac{1}{3}$, and $\frac{1}{3}$ because, unlike the result of tossing the coin, each result is not equally likely to occur.

Variables that might affect the experiment include the dimensions of the index card, the weight of the cardboard, and the angle measure of the fold. (An index card with a 10° opening would be much less likely to form a tent than an index card with a 110° opening.)

## An Empirical Study

Let us go back to the problem of tossing a coin. While we cannot predict the result of one toss of a coin, we can still say that the probability of heads is $\frac{1}{2}$ based on observations made in an empirical study. In an **empirical study**, we perform an experiment many times, keep records of the results, and then analyze these results.

For example, ten students decided to take turns tossing a coin. Each student completed 20 tosses and the number of heads was recorded as shown at the right. If we look at the results and think of the probability of heads as a fraction comparing the number of heads to the total number of tosses, only Maria, with 10 heads out of 20 tosses, had results where the probability was $\frac{10}{20}$, or $\frac{1}{2}$. This fraction is called the **relative frequency**. Elizabeth had the lowest relative frequency of heads, $\frac{6}{20}$. Peter and Debbie tied for the highest relative frequency with $\frac{13}{20}$. Maria's relative frequency was $\frac{10}{20}$ or $\frac{1}{2}$. This does *not* mean that Maria had correct results while the other students were incorrect; the coins simply fell that way.

| | Number of Heads | Number of Tosses |
|---|---|---|
| Albert | 8 | 20 |
| Peter | 13 | 20 |
| Thomas | 12 | 20 |
| Maria | 10 | 20 |
| Elizabeth | 6 | 20 |
| Joanna | 12 | 20 |
| Kathy | 11 | 20 |
| Jeanne | 7 | 20 |
| Debbie | 13 | 20 |
| James | 9 | 20 |

The students decided to combine their results, by expanding the chart, to see what happened when all 200 tosses of the coin were considered. As shown in columns 3 and 4 of the table on the next page, each cumulative result is found by adding all the results up to that point. For example, in the second row, by adding the 8 heads that Albert tossed and the 13 heads that Peter tossed, we find the cumulative number of heads up to this point to be 21, the total number of heads tossed by Albert and Peter together. Similarly, when the 20 tosses that Albert made and the 20 tosses that Peter made are added, the cumulative number of tosses up to this point is 40.

Since each student completed 20 coin tosses, the cumulative number of tosses should increase by 20 for each row. The cumulative number of heads should increase by varying amounts for each row since each student experienced different results.

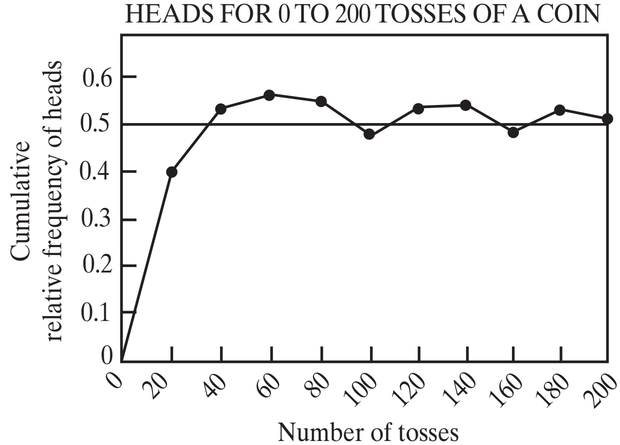
| | (COL. 1) Number of Heads | (COL. 2) Number of Tosses | (COL. 3) Cumulative Number of Heads | (COL. 4) Cumulative Number of Tosses | (COL. 5) Cumulative Relative Frequency |
|---|---|---|---|---|---|
| Albert | 8 | 20 | 8 | 20 | $\frac{8}{20}$ = .400 |
| Peter | 13 | 20 | 21 | 40 | $\frac{21}{40}$ = .525 |
| Thomas | 12 | 20 | 33 | 60 | $\frac{33}{60}$ = .550 |
| Maria | 10 | 20 | 43 | 80 | $\frac{43}{80}$ = .538 |
| Elizabeth | 6 | 20 | 49 | 100 | $\frac{49}{100}$ = .490 |
| Joanna | 12 | 20 | 61 | 120 | $\frac{61}{120}$ = .508 |
| Kathy | 11 | 20 | 72 | 140 | $\frac{72}{140}$ = .514 |
| Jeanne | 7 | 20 | 79 | 160 | $\frac{79}{160}$ = .494 |
| Debbie | 13 | 20 | 92 | 180 | $\frac{92}{180}$ = .511 |
| James | 9 | 20 | 101 | 200 | $\frac{101}{200}$ = .505 |

In column 5, the **cumulative relative frequency** is found by dividing the total number of heads at or above a row by the total number of tosses at or above that row. The cumulative relative frequency is shown as a fraction and then, for easy comparison, as a decimal. The decimal is given to the nearest thousandth.

While the relative frequency for individual students varied greatly, from $\frac{6}{20}$ for Elizabeth to $\frac{13}{20}$ for Peter and Debbie, the cumulative relative frequency, after all 200 tosses were combined, was a number very close to $\frac{1}{2}$.

A graph of the results of columns 4 and 5 will tell us even more. In the graph, the horizontal axis is labeled "Number of tosses" to show the cumulative results of column 4; the vertical axis is labeled "Cumulative relative frequency of heads" to show the results of column 5.

CUMULATIVE RELATIVE FREQUENCY OF HEADS FOR 0 TO 200 TOSSES OF A COIN



On the graph, we have plotted the points that represent the data in columns 4 and 5 of the preceding table, and we have connected these points to form a line graph. Notice how the line moves up and down around the relative frequency of 0.5, or $\frac{1}{2}$. The graph shows that the more times the coin is tossed, the closer the relative frequency comes to $\frac{1}{2}$. In other words, the line seems to level out at a relative frequency of $\frac{1}{2}$. We say that the cumulative relative frequency **converges** to the number $\frac{1}{2}$ and the coin will land heads up about one-half of the time.

Even though the cumulative relative frequency of $\frac{101}{200}$ is not exactly $\frac{1}{2}$, we sense that the line will approach the number $\frac{1}{2}$. When we use carefully collected evidence about tossing a fair coin to guess that the probability of heads is $\frac{1}{2}$, we have arrived at this conclusion empirically, that is, by experimentation and observation.

► ***Empirical probability* may be defined as the most accurate scientific estimate, based on a large number of trials, of the cumulative relative frequency of an event happening.**

## Experiments in Probability

A single attempt at doing something, such as tossing a coin only once, is called a **trial**. We perform **experiments** in probability by repeating the same trial many times. Experiments are aimed at finding the probabilities to be assigned to the occurrences of an event, such as heads coming up on a coin. The objects used in an experiment may be classified into one of two categories:

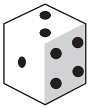
1. **Fair and unbiased objects** have not been weighted or made unbalanced. An object is fair when the different possible results have equal chances of happening. Objects such as coins, cards, and spinners will always be treated in this book as fair objects, unless otherwise noted.
2. **Biased objects** have been tampered with or are weighted to give one result a better chance of happening than another. The folded index card described earlier in this section is a biased object because the probability of each of three results is not $\frac{1}{3}$. The card is weighted so that it will fall on its side more often than it will fall on its edge.

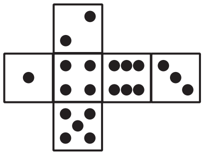
You have seen how to determine empirical probability by the tables and graph previously shown in this section. Sometimes, however, it is possible to guess the probability that should be assigned to the result described before you start an experiment.

In Examples 1–5, use common sense to guess the probability that might be assigned to each result. (The answers are given without comment here. You will learn how to determine these probabilities in the next section.)

## EXAMPLE 1

A **die** is a six-sided solid object (a cube). Each side (or face) is a square. The sides are marked with 1, 2, 3, 4, 5, and 6 pips, respectively. (The plural of *die* is *dice*.) In rolling a fair die, getting a 4 means that the side showing four pips is facing up. Find the probability of getting a 4, or $P(4)$.


Die


Faces of a die

*Answer* $P(4) = \frac{1}{6}$

## EXAMPLE 2

A **standard deck of cards** contains 52 cards. There are four suits: hearts, diamonds, spades, and clubs. Each suit contains 13 cards: 2, 3, 4, 5, 6, 7, 8, 9, 10, jack, queen, king, and ace. The diamonds and hearts are red; the spades and clubs are black. In selecting a card from the deck without looking, find the probability of drawing:

**a.** the 7 of diamonds **b.** a 7 **c.** a diamond

*Answers* **a.** There is only one 7 of diamonds. $P(\text{7 of diamonds}) = \frac{1}{52}$

**b.** There are four 7s. $P(7) = \frac{4}{52}$ or $\frac{1}{13}$

**c.** There are 13 diamonds. $P(\text{diamond}) = \frac{13}{52}$ or $\frac{1}{4}$

## EXAMPLE 3

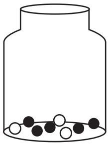
There are 10 digits in our numeral system: 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. In selecting a digit without looking, what is the probability it will be:

**a.** the 8? **b.** an odd digit?

*Answers* **a.** $P(8) = \frac{1}{10}$ **b.** $P(\text{odd}) = \frac{5}{10}$ or $\frac{1}{2}$

## EXAMPLE 4

A jar contains eight marbles: three are white and the remaining five are blue. In selecting a marble without looking, what is the probability it will be blue? (All marbles are the same size.)



*Answer* $P(\text{blue}) = \frac{5}{8}$

**EXAMPLE 5**

The English alphabet contains 26 letters. There are 5 vowels (A, E, I, O, U). The other 21 letters are consonants. If a person turns 26 tiles from a word game face-down and each tile represents a different letter of the alphabet, what is the probability of turning over:

**a.** the A? **b.** a vowel? **c.** a consonant?

*Answers* **a.** $P(\text{A}) = \frac{1}{26}$ **b.** $P(\text{vowel}) = \frac{5}{26}$ **c.** $P(\text{consonant}) = \frac{21}{26}$

## EXERCISES

### Writing About Mathematics

**1.** Alicia read that, in a given year, one out of four people will be involved in an automobile accident. There are four people in Alicia's family. Alicia concluded that this year, one of the people in her family will be involved in an automobile accident. Do you agree with Alicia's conclusion? Explain why or why not.

**2.** A library has a collection of 25,000 books. Is the probability that a particular book will be checked out $\frac{1}{25{,}000}$? Explain why or why not.

### Developing Skills

In 3–8, in each case, a fair, unbiased object is involved. These questions should be answered without conducting an experiment; take a guess.

**3.** The six sides of a number cube are labeled 1, 2, 3, 4, 5, and 6. Find $P(5)$, the probability of getting a 5 when the die is rolled.

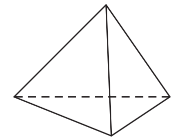
**4.** In drawing a card from a standard deck without looking, find $P(\text{any heart})$.

**5.** Each of 10 pieces of paper contains a different number from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. The pieces of paper are folded and placed in a bag. In selecting a piece of paper without looking, find $P(7)$.

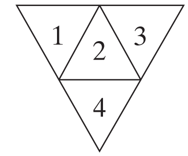
**6.** A jar contains five marbles, all the same size. Two marbles are black, and the other three are white. In selecting a marble without looking, find $P(\text{black})$.

**7.** Using the lettered tiles from a word game, a boy places 26 tiles, one tile for each letter of the alphabet, facedown on a table. After mixing up the tiles, he picks one. What is the probability that the tile contains one of the letters in the word MATH?

8. A *tetrahedron* is a four-sided object. Each side, or face, is an equilateral triangle. The numerals 1, 2, 3, and 4 are used to number the different faces, as shown in the figure. A trial consists of rolling the tetrahedron and reading the number that is face-down. Find $P(4)$.



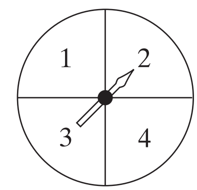
Tetrahedron




Faces of a tetrahedron

9. By yourself, or with some classmates, conduct any of the experiments described in Exercises 3–8 to verify that you have assigned the correct probability to the event or result described. A good experiment should contain at least 100 trials.

10. The figure at the right shows a spinner that has an equal chance of landing on one of four sectors. The regions are equal in size and are numbered 1, 2, 3, and 4.




An experiment was conducted by five people to find the probability that the arrow will land on the 2. Each person spun the arrow 100 times. When the arrow landed on a line, the result did not count and the arrow was spun again.

**a.** Before doing the experiment, what probability would you assign to the arrow landing on the 2? (In symbols, $P(2) = ?$)

**b.** Copy and complete the table below to find the cumulative results of this experiment. In the last column, record the cumulative relative frequencies as fractions and as decimals to the *nearest thousandth*.

| | Number of Times Arrow Landed on 2 | Number of Spins | Cumulative Number of Times Arrow Landed on 2 | Cumulative Number of Spins | Cumulative Relative Frequency |
|---|---|---|---|---|---|
| Barbara | 29 | 100 | 29 | 100 | $\frac{29}{100} = .290$ |
| Tom | 31 | 100 | 60 | 200 | |
| Ann | 19 | 100 | | | |
| Eddie | 23 | 100 | | | |
| Cathy | 24 | 100 | | | |

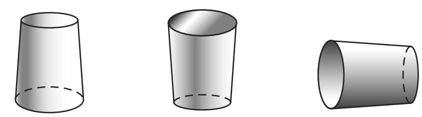
**c.** Did the experiment provide evidence that the probability you assigned in part **a** was correct?

**d.** Form a group of five people and repeat the experiment, making a table similar to the one shown above. Do your results provide evidence that the probability you assigned in **a** was correct?

In 11–15, a biased object is described. A probability can be assigned to a result only by conducting an experiment to determine the cumulative relative frequency of the event. While you may wish to guess at the probability of the event before starting the experiment, conduct at least 100 trials to determine the best probability to be assigned.

**11.** An index card is folded in half and tossed. As described earlier in this section, the card may land in one of three positions: on its side, on its edge, or in the form of a tent. In tossing the folded card, find $P$(tent), the probability that the card will form a tent when it lands.

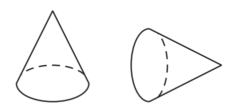
**12.** A paper cup is tossed. It can land in one of three positions: on its top, on its bottom, or on its side, as shown in the figure. In tossing the cup, find $P$(top), the probability that the cup will land on its top.



Top Bottom Side

**13.** A nickel and a quarter are glued or taped together so that the two faces seen are the head of the quarter and the tail of the nickel. (This is a very crude model of a weighted coin.) In tossing the coin, find $P$(head).

**14.** A paper cup in the shape of a cone is tossed. It can land in one of two positions: on its base or on its side, as shown in the figure. In tossing this cup, find $P$(side), the probability that the cup will land on its side.



Base Side

**15.** A thumbtack is tossed. It may land either with the pin up or with the pin down, as shown in the figure. In tossing the thumbtack, find $P$(pin up).

Pin up Pin down

**16.** The first word is selected from a page of a book written in English.

**a.** What is the probability that the word contains at least one of the letters $a$, $e$, $i$, $o$, $u$, or $y$?

**b.** In general, what is the largest possible probability?

**c.** What is the probability that the word does *not* contain at least one of the letters $a$, $e$, $i$, $o$, $u$, or $y$?

**d.** In general, what is the smallest possible probability?

## Applying Skills

**17.** An insurance company's records show that last year, of the 1,000 cars insured by the company, 210 were involved in accidents. What is the probability that an insurance policy, chosen at random from their files, is that of a car that was *not* involved in an accident?

**18.** A school's attendance records for last year show that of the 885 students enrolled, 15 had no absences for the year. What is the probability that a student, chosen at random, had no absences?

**19.** A chess club consists of 45 members of whom 24 are boys and 21 are girls. If a member of the club is chosen at random to represent the club at a tournament, what is the probability that the person chosen is a boy?

## 15-2 THEORETICAL PROBABILITY

An empirical approach to probability is necessary whenever we deal with biased objects. However, common sense tells us that there is a simple way to define the probability of an event when we deal with fair, unbiased objects.

For example, let us suppose that Alma is playing a game in which each player must roll a die. To win, Alma must roll a number greater than 4. What is the probability that Alma will win on her next turn? Common sense tells us that:

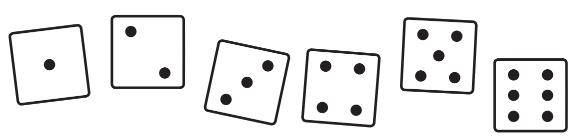
1. The die has an equal chance of falling in any one of *six* ways: 1, 2, 3, 4, 5, and 6.
2. There are *two* ways for Alma to win: rolling a 5 or a 6.
3. Therefore: $P(\text{Alma wins}) = \frac{\text{number of winning results}}{\text{number of possible results}} = \frac{2}{6} = \frac{1}{3}$.

### Terms and Definitions

Using the details of the preceding example, let us examine the correct terminology to be used.

An **outcome** is a result of some activity or experiment. In rolling a die, 1 is an outcome, 2 is an outcome, 3 is an outcome, and so on. There are six outcomes when rolling a six-sided die.

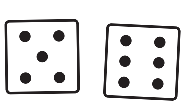
A **sample space** is a set of all possible outcomes for the activity. When rolling a die, there are six possible outcomes in the sample space: 1, 2, 3, 4, 5, and 6. We say that the sample space is {1, 2, 3, 4, 5, 6}.



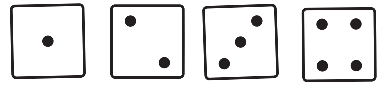
An **event** is a subset of the sample space. We use the term *event* in two ways. In ordinary conversation, it means a situation or happening. In the technical language of probability, it is the subset of the sample space that lists all of the outcomes for a given situation.

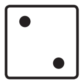
When we focus on a particular event, such as heads facing up when we toss a coin, we refer to it as the **favorable event**. The other event or events in the sample space, in this case tails, are called **unfavorable**.

When we roll a die, we may define many different situations. Each of these is called an event.



1. For Alma, the event of rolling a number *greater than* 4 contains only two outcomes: 5 and 6.
2. For Lee, a different event might be rolling a number *less than* 5. This event contains four outcomes: 1, 2, 3, and 4.





**3.** For Sandi, the event of rolling a 2 contains only one outcome: 2. When there is only one outcome, we call this a **singleton event**.

We can now define **theoretical probability** for fair, unbiased objects:

► **The theoretical probability of an event is the number of ways that the event can occur, divided by the total number of possibilities in the sample space.**

In symbolic form, we write:

$$\boldsymbol{P(E) = \frac{n(E)}{n(S)}}$$

where

$P(E)$ represents the probability of event $E$;
$n(E)$ represents the number of ways event $E$ can occur or the number of outcomes in event $E$;
$n(S)$ represents the total number of possibilities, or the number of outcomes in sample space $S$.

Since theoretical probability relies on calculation as opposed to experimentation, it is sometimes referred to as **calculated probability**.

Let us now use this formula to write the probabilities of the three events described above.

**1.** For Alma, there are two ways to roll a number greater than 4, and there are six possible ways that the die may fall. We say:

$E$ = the set of numbers on a die that are greater than 4: {5, 6}.
$n(E)$ = 2 since there are 2 outcomes in this event.
$S$ = the set of all possible outcomes: {1, 2, 3, 4, 5, 6}.
$n(S)$ = 6 since there are six outcomes in the sample space.

Therefore:

$$P(E) = \frac{n(E)}{n(S)} = \frac{2}{6} = \frac{1}{3}$$

**2.** For Lee, the probability of rolling a number less than 5 is

$$P(E) = \frac{n(E)}{n(S)} = \frac{4}{6} = \frac{2}{3}$$

**3.** For Sandi, the probability of rolling a 2 is

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{6}$$

## Uniform Probability

A sample space is said to have **uniform probability**, or to contain **equally likely outcomes**, when each of the possible outcomes has an equal chance of occurring. In rolling a die, there are six possible outcomes in the sample space; each is equally likely to occur. Therefore:

$$P(1) = \tfrac{1}{6}; P(2) = \tfrac{1}{6}; P(3) = \tfrac{1}{6}; P(4) = \tfrac{1}{6}; P(5) = \tfrac{1}{6}; P(6) = \tfrac{1}{6}$$

We say that the die has uniform probability.

If, however, a die is weighted to make it biased, then one or more sides will have a probability greater than $\frac{1}{6}$, while one or more sides will have a probability less than $\frac{1}{6}$. A weighted die does not have uniform probability; the rule for theoretical probability does *not* apply to weighted objects.

## Random Selection

When we select an object from a collection of objects without knowing any of the special characteristics of the object, we are making a **random selection**. Random selections are made when drawing a marble from a bag, when taking a card from a deck, or when picking a name out of a hat. In the same way, we may use the word *random* to describe the outcomes when tossing a coin or rolling a die; the outcomes happen without any special selection on our part.

### Procedure

**To find the simple probability of an event:**

1. Count the total number of outcomes in the sample space $S$: $n(S)$.
2. Count all the possible outcomes of the event $E$: $n(E)$.
3. Substitute these values in the formula for the probability of event $E$:

$$P(E) = \frac{n(E)}{n(S)}$$

**Note:** The probability of an event is usually written as a fraction. A standard calculator display, however, is in decimal form. Therefore, if you use a calculator when working with probability, it is helpful to know fraction-decimal equivalents.

Recall that some common fractions have equivalent decimals that are terminating decimals:

$$\tfrac{1}{2} = 0.5 \qquad \tfrac{1}{4} = 0.25 \qquad \tfrac{1}{5} = 0.2 \qquad \tfrac{1}{8} = 0.125$$

Others have equivalent decimals that are repeating decimals:

$$\tfrac{1}{3} = 0.\overline{3} \qquad \tfrac{2}{3} = 0.\overline{6} \qquad \tfrac{1}{6} = 0.1\overline{6} \qquad \tfrac{1}{9} = 0.\overline{1}$$

The graphing calculator has a function that will change a decimal fraction to a common fraction in lowest terms. For example, the probability of a die showing a number greater than 4 can be displayed on a calculator as follows:

ENTER: 2 **÷** 6 **ENTER**

DISPLAY:
```
2/6
          .3333333333
```

ENTER: **MATH** **1** **ENTER**

DISPLAY:
```
2/6
          .3333333333
Ans▶Frac
                  1/3
```

## EXAMPLE 1

A standard deck of 52 cards is shuffled. Daniella draws a single card from the deck at random. What is the probability that the card is a jack?

***Solution*** $S$ = sample space of all possible outcomes, or 52 cards. Thus, $n(S) = 52$.
$J$ = event of selecting a jack. There are four jacks in the deck: jack of hearts, of diamonds, of spades, and of clubs. Thus, $n(J) = 4$.

$$P(J) = \frac{\text{number of possible jacks}}{\text{number of possible cards}} = \frac{n(J)}{n(S)} = \frac{4}{52} = \frac{1}{13}$$ *Answer*

***Calculator Solution*** ENTER: 4 **÷** 52 **MATH** **1** **ENTER**

DISPLAY:
```
4/52▶Frac
                 1/13
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## EXAMPLE 2

An aquarium at a pet store contains 8 goldfish, 6 angelfish, and 7 zebrafish. David randomly chooses a fish to take home for his fishbowl.

**a.** How many possible outcomes are in the sample space?

**b.** What is the probability that David takes home a zebrafish?

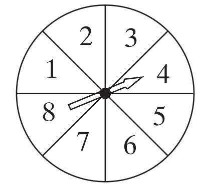
***Solution*** **a.** Each fish represents a distinct outcome. Therefore, if $S$ is the sample space of all possible outcomes, then $n(S) = 8 + 6 + 7 = 21$.

**b.** Since there are 7 zebrafish: $P(\text{zebrafish}) = \frac{7}{21} = \frac{1}{3}$.

*Answers* **a.** 21 **b.** $\frac{1}{3}$

### EXAMPLE 3

A spinner contains eight regions, numbered 1 through 8, as shown in the figure. The arrow has an equally likely chance of landing on any of the eight regions. If the arrow lands on a line, the result is not counted and the arrow is spun again.




**a.** How many possible outcomes are in the sample space $S$?

**b.** What is the probability that the arrow lands on the 4? That is, what is $P(4)$?

**c.** List the set of possible outcomes for event $O$, in which the arrow lands on an odd number.

**d.** Find the probability that the arrow lands on an odd number.

*Answers*

**a.** $n(S) = 8$

**b.** Since there is only one region numbered 4 out of eight regions: $P(4) = \frac{1}{8}$

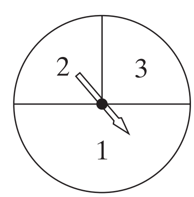
**c.** Event $O = \{1, 3, 5, 7\}$

**d.** Since $O = \{1, 3, 5, 7\}$, $n(O) = 4$. $P(O) = \frac{n(O)}{n(S)} = \frac{4}{8} = \frac{1}{2}$

## EXERCISES

### Writing About Mathematics

**1.** A spinner is divided into three sections numbered 1, 2, and 3, as shown in the figure. Explain why the probability of the arrow landing on the region numbered 1 is not $\frac{1}{3}$.




**2.** Mark said that since there are 50 states, the probability that the next baby born in the United States will be born in New Jersey is $\frac{1}{50}$. Do you agree with Mark? Explain why or why not.

### Developing Skills

**3.** A fair coin is tossed.

**a.** List the sample space.

**b.** What is $P(\text{head})$, the probability that a head will appear?

**c.** What is $P(\text{tail})$?

In 4–9, a fair die is tossed. For each question: **a**. List the possible outcomes for the event. **b.** State the probability of the event.

**4.** The number 3 appears.

**5.** An even number appears.

**6.** A number less than 3 appears.

**7.** An odd number appears.

**8.** A number greater than 3 appears.

**9.** A number greater than or equal to 3 appears.

In 10–15, a spinner is divided into five equal regions, numbered 1 through 5, as shown below. The arrow is spun and lands in one of the regions. For each question:

**a.** List the outcomes for the event. **b.** State the probability of the event.

**10.** The arrow lands on number 3.

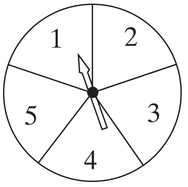
**11.** The arrow lands on an even number.

**12.** The arrow lands on a number less than 3.

**13.** The arrow lands on an odd number.

**14.** The arrow lands on a number greater than 3.

**15.** The arrow lands on a number greater than or equal to 3.




**16.** A standard deck of 52 cards is shuffled, and one card is drawn. What is the probability that the card is:

**a.** the queen of hearts? **b.** a queen? **c.** a heart?
**d.** a red card? **e.** the 7 of clubs? **f.** a club?
**g.** an ace? **h.** a red 7? **i.** a black 10?
**j.** a picture card (king, queen, jack)?

**17.** A person does not know the answer to a test question and takes a guess. Find the probability that the answer is correct if the question is:

**a.** a multiple-choice question with four choices

**b.** a true-false question

**c.** a question where the choices given are "sometimes, always, or never"

**18.** A marble is drawn at random from a bag. Find the probability that the marble is green if the bag contains marbles whose colors are:

**a.** 3 blue, 2 green **b.** 4 blue, 1 green **c.** 5 red, 2 green, 3 blue
**d.** 6 blue, 4 green **e.** 3 green, 9 blue **f.** 5 red, 2 green, 9 blue

**19.** The digits of the number 1,776 are written on disks and placed in a jar. What is the probability that the digit 7 will be chosen on a single random draw?

**20.** A letter is chosen at random from a given word. Find the probability that the letter is a vowel if the word is:

**a.** APPLE **b.** BANANA **c.** GEOMETRY **d.** MATHEMATICS

## Applying Skills

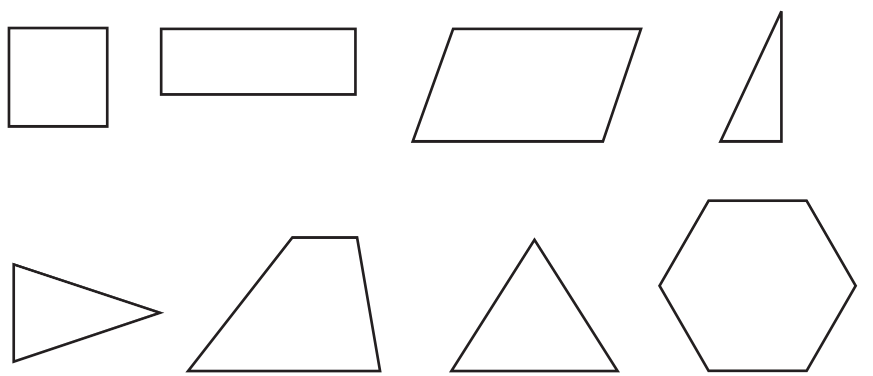
**21.** There are 16 boys and 14 girls in a class. The teacher calls students at random to the chalkboard. What is the probability that the first person called is:

**a.** a boy? **b.** a girl?

**22.** There are 840 tickets sold in a raffle. Jay bought five tickets, and Lynn bought four tickets. What is the probability that:

**a.** Jay has the winning ticket? **b.** Lynn has the winning ticket?

**23.** The figures that follow are eight polygons: a square; a rectangle; a parallelogram that is not a rectangle; a right triangle; an isosceles triangle that does not contain a right angle; a trapezoid that does not contain a right angle; an equilateral triangle; a regular hexagon.



One of the figures is selected at random. What is the probability that this polygon:

**a.** contains a right angle? **b.** is a quadrilateral?

**c.** is a triangle? **d.** has at least one acute angle?

**e.** has all sides congruent? **f.** has at least two sides congruent?

**g.** has fewer than five sides? **h.** has an odd number of sides?

**i.** has four or more sides? **j.** has at least two obtuse angles?

## 15-3 EVALUATING SIMPLE PROBABILITIES

We have called an event for which there is only one outcome a singleton. For example, when rolling a die only once, getting a 3 is a singleton.

However, when rolling a die only once, some events are not singletons. For example:

**1.** The event of rolling an even number on a die is {2, 4, 6}.

**2.** The event of rolling a number less than 6 on a die is {1, 2, 3, 4, 5}.

## The Impossible Case

On a single roll of a die, what is the probability that the number 7 will appear? We call this case an **impossibility** because there is no way in which this event can occur. In this example, event $E$ = rolling a 7; so, $E = \{\,\}$ or $\varnothing$, and $n(E) = 0$. The sample space $S$ for rolling a die contains six possible outcomes, and $n(S) = 6$. Therefore:

$$P(E) = \frac{\text{number of ways to roll a 7}}{\text{number of outcomes for the die}} = \frac{n(E)}{n(S)} = \frac{0}{6} = 0$$

In general, for any sample space $S$ containing $k$ possible outcomes, we say $n(S) = k$. For any impossible event $E$, which cannot occur in any way, we say $n(E) = 0$. Thus, the probability of an impossible event is:

$$\mathbf{P(E) = \frac{n(E)}{n(S)} = \frac{0}{k} = 0}$$

and we say:

► **The probability of an impossible event is 0.**

There are many other impossibilities where the probability must equal zero. For example, the probability of selecting the letter $E$ from the word PROBABILITY is 0. Also, selecting a coin worth 9 cents from a bank containing a nickel, a dime, and a quarter is an impossible event.

## The Certain Case

On a single roll of a die, what is the probability that a whole number less than 7 will appear? We call this case a **certainty** because every one of the possible outcomes in the sample space is also an outcome for this event. In this example, the event $E$ = rolling a whole number less than 7, so $n(E) = 6$. The sample space $S$ for rolling a die contains six possible outcomes, so $n(S) = 6$. Therefore:

$$P(E) = \frac{\text{number of ways to roll a number less than 7}}{\text{number of outcomes for the die}} = \frac{n(E)}{n(S)} = \frac{6}{6} = 1$$

When an event $E$ is certain, the event $E$ is the same as the sample space $S$, that is, $E = S$ and $n(E) = n(S)$.

In general, for any sample space $S$ containing $k$ possible outcomes, $n(S) = k$. When the event $E$ is certain, every possible outcome for the sample space is also an outcome for event $E$, or $n(E) = k$. Thus, the probability of a certainty is given as:

$$\mathbf{P(E) = \frac{n(E)}{n(S)} = \frac{k}{k} = 1}$$

and we say:

► **The probability of an event that is certain to occur is 1.**

There are many other certainties where the probability must equal 1. Examples include the probability of selecting a consonant from the letters JFK or selecting a red sweater from a drawer containing only red sweaters.

## The Probability of Any Event

The smallest possible probability is 0, for an impossible case; no probability can be less than 0. The largest possible probability is 1, for a certain event; no probability can be greater than 1. Many other events, as seen earlier, however, have probabilities that fall between 0 and 1. Therefore, we conclude:

► **The probability of any event *E* must be equal to or greater than 0, and less than or equal to 1:**

$$0 \le P(E) \le 1$$

## Subscripts in Sample Spaces

A sample space may sometimes contain two or more objects that are exactly alike. To distinguish one object from another, we make use of subscripts. A **subscript** is a number, usually written in smaller size to the lower right of a term.

For example, a box contains six jellybeans: two red, three green, and one yellow. Using $R$, $G$, and $Y$ to represent the colors red, green, and yellow, respectively, we can list this sample space in full, using subscripts:

$$\{R_1, R_2, G_1, G_2, G_3, Y_1\}$$

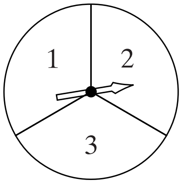
Since there is only one yellow jellybean, we could have listed the last outcome as $Y$ instead of $Y_1$.

**EXAMPLE 1**

An arrow is spun once and lands on one of three equally likely regions, numbered 1, 2, and 3, as shown in the figure.

**a.** List the sample space for this experiment.

**b.** List all eight possible events for one spin of the arrow.




***Solution*** **a.** The sample space $S = \{1, 2, 3\}$.

**b.** Since events are subsets of the sample space $S$, the eight possible events are the eight subsets of $S$:

{ }, the empty set for *impossible* events.
The arrow lands on a number other than 1, 2, or 3.

{1}, a *singleton*.
The arrow lands on 1 or the arrow lands on a number less than 2.

{2}, a *singleton*.
The arrow lands on 2 or the arrow lands on an even number.

{3}, a *singleton*.
The arrow lands on 3 or the arrow lands on a number greater than 2.

{1, 2}, an event with two possible outcomes.
The arrow does not land on 3 or does land on a number less than 3.

{1, 3}, an event with two possible outcomes.
The arrow lands on an odd number or does not land on 2.

{2, 3}, an event with two possible outcomes.
The arrow lands on a number greater than 1 or does not land on 1.

{1, 2, 3}, the sample space itself for events that are certain.
The arrow lands on a whole number less than 4 or greater than 0.

*Answers* **a.** $S = \{1, 2, 3\}$ **b.** { }, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}

**EXAMPLE 2**

A piggy bank contains a nickel, two dimes, and a quarter. A person selects one of the coins. What is the probability that the coin is worth:

**a.** exactly 10 cents? **b.** at least 10 cents?

**c.** exactly 3 cents? **d.** more than 3 cents?

***Solution*** The sample space for this example is $\{N, D_1, D_2, Q\}$. Therefore, $n(S) = 4$.

**a.** There are two coins worth exactly 10 cents, $D_1$ and $D_2$. Therefore, $n(E) = 2$ and

$$P(\text{coin worth 10 cents}) = \frac{n(E)}{n(S)} = \frac{2}{4} = \frac{1}{2}.$$

**b.** There are three coins worth at least 10 cents, $D_1$, $D_2$, and $Q$. Therefore, $n(E) = 3$ and

$$P(\text{coin worth at least 10 cents}) = \frac{n(E)}{n(S)} = \frac{3}{4}.$$

**c.** There is no coin worth exactly 3 cents. This is an impossible event. Therefore,

$$P(\text{coin worth 3 cents}) = 0.$$

**d.** Each of the four coins is worth more than 3 cents. This is a certain event. Therefore,

$$P(\text{coin worth more than 3 cents}) = 1.$$

*Answers* **a.** $\frac{1}{2}$ **b.** $\frac{3}{4}$ **c.** 0 **d.** 1

**EXAMPLE 3**

In the Sullivan family, there are two more girls than boys. At random, Mrs. Sullivan asks one of her children to go to the store. If she is equally likely to ask any one of her children, and the probability that she asks a girl is $\frac{2}{3}$, how many boys and how many girls are there in the Sullivan family?

***Solution*** Let $x =$ the number of boys

$x + 2 =$ the number of girls

$2x + 2 =$ the number of children.

Then:

$$P(\text{girl}) = \frac{\text{number of girls}}{\text{number of children}}$$
$$\frac{2}{3} = \frac{x + 2}{2x + 2}$$
$$2(2x + 2) = 3(x + 2)$$
$$4x + 4 = 3x + 6$$
$$x = 2$$
$$\text{Then } x + 2 = 4$$
$$\text{and } 2x + 2 = 6$$

*Check*

$$P(\text{girl}) = \frac{\text{number of girls}}{\text{number of children}}$$
$$\frac{2}{3} \stackrel{?}{=} \frac{4}{6}$$
$$\frac{2}{3} = \frac{2}{3} \checkmark$$

*Answer* There are two boys and four girls.

## EXERCISES

### Writing About Mathematics

**1.** Describe three events for which the probability is 0.

**2.** Describe three events for which the probability is 1.

### Developing Skills

**3.** A fair coin is tossed, and its sample space is $S = \{H, T\}$.

**a.** List all four possible events for the toss of a fair coin.

**b.** Find the probability of each event named in part **a**.

In 4–11, a spinner is divided into seven equal sectors, numbered 1 through 7. An arrow is spun to fall into one of the regions. For each question, find the probability that the arrow lands on the number described.

**4.** the number 5

**5.** an even number

**6.** a number less than 5

**7.** an odd number

**8.** a number greater than 5

**9.** a number greater than 1

**10.** a number greater than 7

**11.** a number less than 8

**12.** A marble is drawn at random from a bag. Find the probability that the marble is black if the bag contains marbles whose colors are:

**a.** 5 black, 2 green **b.** 2 black, 1 green **c.** 3 black, 4 green, 1 red

**d.** 9 black **e.** 3 green, 4 red **f.** 3 green

**13.** Ted has two quarters, three dimes, and one nickel in his pocket. He pulls out a coin at random. Find the probability that the coin is worth:

**a.** exactly 5 cents **b.** exactly 10 cents **c.** exactly 25 cents

**d.** exactly 50 cents **e.** less than 25 cents **f.** less than 50 cents

**g.** more than 25 cents **h.** more than l cent **i.** less than l cent

**14.** A single fair die is rolled. Find the probability for each event.

**a.** The number 8 appears. **b.** A whole number appears.

**c.** The number is less than 5. **d.** The number is less than 1.

**e.** The number is less than 10. **f.** The number is negative.

**15.** A standard deck of 52 cards is shuffled, and a card is picked at random. Find the probability that the card is:

**a.** a jack **b.** a club **c.** a star

**d.** a red club **e.** a card from the deck **f.** a black club

**g.** the jack of stars **h.** a 17 **i.** a red 17

In 16–20, a letter is chosen at random from a given word. For each question: **a.** Write the sample space, using subscripts to designate events if needed. **b.** Find the probability of the event.

**16.** Selecting the letter E from the word EVENT

**17.** Selecting the letter S from the word MISSISSIPPI

**18.** Selecting a vowel from the word TRIANGLE

**19.** Selecting a vowel from the word RECEIVE

**20.** Selecting a consonant from the word SPRY

## Applying Skills

**21.** There are 15 girls and 10 boys in a class. The teacher calls on a student in the class at random to answer a question. Express, *in decimal form*, the probability that the student called upon is:

**a.** a girl **b.** a boy **c.** a pupil in the class

**d.** a person who is not a student in the class

**22.** The last digit of a telephone number can be any of the following: 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9. Express, *as a percent*, the probability that the last digit is:

**a.** 7 **b.** odd **c.** greater than 5 **d.** a whole number **e.** the letter R

**23.** A girl is holding five cards in her hand: the 3 of hearts, the 3 of diamonds, the 3 of clubs, the 4 of diamonds, the 7 of clubs. A player to her left takes one of these cards at random. Find the probability that the card selected from the five cards in the girl's hand is:

**a.** a 3 **b.** a diamond **c.** a 4
**d.** a black 4 **e.** a club **f.** the 4 of hearts
**g.** a 5 **h.** the 7 of clubs **i.** a red card
**j.** a number card **k.** a spade **l.** a number greater than 1 and less than 8.

**24.** A sack contains 20 marbles. The probability of drawing a green marble is $\frac{2}{5}$. How many green marbles are in the sack?

**25.** There are three more boys than girls in the chess club. A member of the club is to be chosen at random to play in a tournament. Each member is equally likely to be chosen. If the probability that a girl is chosen is $\frac{3}{7}$, how many boys and how many girls are members of the club?

**26.** A box of candy contains caramels and nut clusters. There are six more caramels than nut clusters. If a piece of candy is to be chosen at random, the probability that it will be a caramel is $\frac{3}{5}$. How many caramels and how many nut clusters are in the box?

**27.** At a fair, each ride costs one, two, or four tickets. The number of rides that cost two tickets is three times the number of rides that cost one ticket. Also, seven more rides cost four tickets than cost two tickets. Tycella, who has a book of tickets, goes on a ride at random. If the probability that the ride cost her four tickets is $\frac{4}{7}$, how many rides are there at the fair?

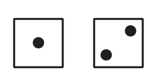
## 15-4 THE PROBABILITY OF (*A* AND *B*)

If a fair die is rolled, we can find simple probabilities, since we know that $S = \{1, 2, 3, 4, 5, 6\}$. For example, let event $A$ be rolling an even number. Then $A = \{2, 4, 6\}$ and

$$P(A) = \frac{n(A)}{n(S)} = \frac{3}{6}$$

Let event $B$ be rolling a number less than 3. Then $B = \{1, 2\}$ and

$$P(B) = \frac{n(B)}{n(S)} = \frac{2}{6}$$



Now, what is the probability of obtaining a number on the die that is even and less than 3? We may think of this as event (*A and B*), which consists of those elements of the sample space that are in $A$ and also in $B$. In set notation we say,

$$(A \text{ and } B) = A \cap B = \{2\}$$

Only 2 is both even and less than 3. Notice that we use the symbol for intersection ($\cap$) to denote *and*. Since $n(A \text{ and } B) = 1$,

$$P(A \text{ and } B) = \frac{n(A \text{ and } B)}{n(S)} = \frac{1}{6}$$

Let us consider another example in which a fair die is rolled. What is the probability of rolling a number that is both odd *and* a 4?

Event $C = \{1, 3, 5\}$. Three numbers on a die are odd.

$$P(C) = \frac{n(C)}{n(S)} = \frac{3}{6}$$

Event $D = \{4\}$. One number on the die is 4.

$$P(D) = \frac{n(D)}{n(S)} = \frac{1}{6}$$

$(C \text{ and } D) = \{$numbers on a die that are odd *and* 4$\} = C \cap D = \varnothing$, the empty set.

Since there is *no* outcome common to both $C$ and $D$, $n(C \text{ and } D) = 0$. Therefore,

$$P(C \cap D) = \frac{n(C \cap D)}{n(S)} = \frac{0}{6} = 0$$

## Conclusions

There is no simple rule or formula whereby the $n(A)$ and $n(B)$ can be used to find $n(A \text{ and } B)$. We must simply count the number of outcomes that are common in both events or write the intersection of the two sets and count the number of elements in that intersection.

**KEEP IN MIND** Event ($A$ and $B$) consists of the outcomes that are in event $A$ *and* in event $B$. Event ($A$ and $B$) may be regarded as the *intersection* of sets, namely, $A \cap B$.

### EXAMPLE 1

A fair die is rolled once. Find the probability of obtaining a number that is greater than 3 and less than 6.

***Solution*** Event $A = \{$numbers greater than 3$\} = \{4, 5, 6\}$.

Event $B = \{$numbers less than 6$\} = \{1, 2, 3, 4, 5\}$.

Event $(A \text{ and } B) = \{$numbers greater than 3 *and* less than 6$\} = \{4, 5\}$.

Therefore: $P(A \text{ and } B) = \frac{n(A \text{ and } B)}{n(S)} = \frac{2}{6}$ or $P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{2}{6}$.

*Answer* $P(\text{number greater than 3 and less than 6}) = \frac{2}{6}$ or $\frac{1}{3}$

## EXERCISES

### Writing About Mathematics

**1.** Give an example of two events $A$ and $B$, such that $P(A \text{ and } B) = P(A)$.

**2.** If $P(A \text{ and } B) = P(A)$, what must be the relationship between set $A$ and set $B$?

### Developing Skills

**3.** A fair die is rolled once. The sides are numbered 1, 2, 3, 4, 5, and 6. Find the probability that the number rolled is:

**a.** greater than 2 and odd
**b.** less than 4 and even
**c.** greater than 2 and less than 4
**d.** less than 2 and even
**e.** less than 6 and odd
**f.** less than 4 and greater than 3

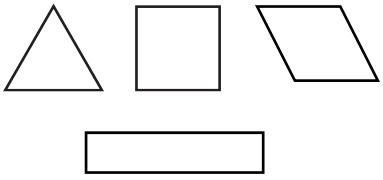
**4.** From a standard deck of cards, one card is drawn. Find the probability that the card is:

**a.** the king of hearts
**b.** a red king
**c.** a king of clubs
**d.** a black jack
**e.** a 10 of diamonds
**f.** a red club
**g.** a 2 of spades
**h.** a black 2
**i.** a red picture card

### Applying Skills

**5.** A set of polygons consists of an equilateral triangle, a square, a rhombus that is not a square, and a rectangle. One of the polygons is selected at random.

Find the probability that the polygon contains:



**a.** all sides congruent and all angles congruent
**b.** all sides congruent and all right angles
**c.** all sides congruent and two angles not congruent
**d.** at least two congruent sides and at least two congruent angles
**e.** at least three congruent sides and at least two congruent angles

**6.** In a class of 30 students, 23 take science, 28 take math, and all take either science or math.

**a.** How many students take both science *and* math?
**b.** A student from the class is selected at random. Find:

(1) $P$(takes science) (2) $P$(takes math) (3) $P$(takes science *and* math)

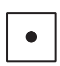
**7.** At a karaoke party, some of the boys and girls take turns singing songs. Of the five boys, Patrick and Terence are teenagers while Brendan, Drew, and Kevin are younger. Of the seven girls, Heather and Claudia are teenagers while Maureen, Elizabeth, Gwen, Caitlin, and Kelly are younger. Find the probability that the first song is sung by:

**a.** a girl
**b.** a boy
**c.** a teenager
**d.** someone under 13 years old
**e.** a boy under 13
**f.** a girl whose initial is C
**g.** a teenage girl
**h.** a girl under 13
**i.** a boy whose initial is C
**j.** a teenage boy

## 15-5 THE PROBABILITY OF (*A* OR *B*)

If a fair die is rolled, we can find simple probabilities. For example, let event $A$ be rolling an even number. Then:

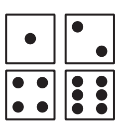
$$P(A) = \frac{n(A)}{n(S)} = \frac{3}{6}$$



Let event $C$ be rolling a number less than 2. Then:

$$P(C) = \frac{n(C)}{n(S)} = \frac{1}{6}$$

Now, what is the probability of obtaining a number on the die that is even or less than 2? We may think of this as event ($A$ *or* $C$).



For the example above, there are four outcomes in the event ($A$ or $C$): 1, 2, 4, and 6. Each of these numbers is either even or less than 2 or both. Since $n(A \text{ or } C) = 4$ and there are six elements in the sample space:

$$P(A \text{ or } C) = \frac{n(A \text{ or } C)}{n(S)} = \frac{4}{6}$$

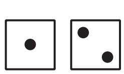
Observe that $P(A) = \frac{3}{6}$, $P(C) = \frac{1}{6}$ and $P(A \text{ or } C) = \frac{4}{6}$. In this case, it appears that

$$P(A) + P(C) = P(A \text{ or } C).$$

Will this simple addition rule hold true for all problems? Before you say "yes," consider the next example, in which a fair die is rolled.

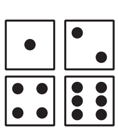
Event $A$ = {even numbers on a die} = {2, 4, 6}.

$$P(A) = \frac{n(A)}{n(S)} = \frac{3}{6}$$



Event $B$ = {numbers less than 3 on a die} = {1, 2}.

$$P(B) = \frac{n(B)}{n(S)} = \frac{2}{6}$$



Then, event ($A$ or $B$) = {numbers that are even or less than 3}.

$$P(A \text{ or } B) = \frac{n(A \text{ or } B)}{n(S)} = \frac{4}{6}$$

Here $P(A) = \frac{3}{6}$, $P(B) = \frac{2}{6}$, and $P(A \text{ or } B) = \frac{4}{6}$. In this case, the simple rule of addition does not work: $P(A) + P(B) \neq P(A \text{ or } B)$. What makes this example different from $P(A \text{ or } C)$, shown previously?

## A Rule for the Probability of (*A* or *B*)

Probability is based on the number of outcomes in a given event. For the event ($A$ or $B$) in our example, we observe that the outcome 2 is found in event $A$ *and* in event $B$. Therefore, we may describe rolling a 2 as the event ($A$ and $B$).

The simple addition rule does not work for the event ($A$ or $B$) because we have counted the event ($A$ and $B$) twice: first in event $A$, then again in event $B$. In order to count the event ($A$ and $B$) only once, we must subtract the number of shared elements, $n(A \text{ and } B)$, from the overall number of elements in ($A$ or $B$).

Thus, the rule becomes: $n(A \text{ or } B) = n(A) + n(B) - n(A \text{ and } B)$

For this example: $n(A \text{ or } B) = 3 + 2 - 1 = 4$

Dividing each term by $n(S)$, we get an equivalent equation:

For this example:
$$\frac{n(A \text{ or } B)}{n(S)} = \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} - \frac{n(A \text{ and } B)}{n(S)}$$
$$\frac{n(A \text{ or } B)}{n(S)} = \frac{3}{6} + \frac{2}{6} - \frac{1}{6} = \frac{4}{6}$$

Since $P(A \text{ or } B) = \frac{n(A \text{ or } B)}{n(S)}$, we can write a general rule:

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

In set terminology, the rule for probability becomes:

$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Note the use of the union symbol ($\cup$) to indicate *or*.

## Mutually Exclusive Events

We have been examining three different events that occur when a die is tossed.

Event $A = \{\text{an even number}\} = \{2, 4, 6\}$

Event $B = \{\text{a number less than 3}\} = \{1, 2\}$

Event $C = \{\text{a number less than 2}\} = \{1\}$

We found that

$$P(A \text{ or } C) = P(A) + P(C)$$
$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Why are these results different?

Of these sets, $A$ and $C$ are *disjoint*, that is they have no element in common. Events $A$ and $C$ are said to be **mutually exclusive events** because only one of the events can occur at any one throw of the dice. Events that are disjoint sets are mutually exclusive.

If two events $A$ and $C$ are mutually exclusive:

$$\mathbf{P(A \text{ or } C) = P(A) + P(C)}$$

Sets $A$ and $B$ are *not* disjoint sets. They have element 2 in common. Events $A$ and $B$ are *not* mutually exclusive events because both can occur at any one throw of the dice. Events that are not disjoint sets are not mutually exclusive.

If two events $A$ and $B$ are not mutually exclusive:

$$\mathbf{P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)}$$

Some examples of mutually exclusive events include the following.

- Drawing a spade or a red card when one card is drawn from a standard deck.
- Choosing a ninth-grade boy or a tenth-grade girl from a student body that consists of boys and girls in each grade 9 through 12.
- Drawing a quarter or a dime when one coin is drawn from a purse that contains three quarters, two dimes, and a nickel.
- Drawing a consonant or a vowel when a letter is drawn from the word PROBABILITY.

Some examples of events that are not mutually exclusive include the following.

- Drawing an ace or a red card when one card is drawn from a standard deck.
- Choosing a girl or a ninth-grade student from a student body that consists of boys and girls in each grade 9 through 12.
- Drawing a dime or a coin worth more than five cents when one coin is drawn from a purse that contains a quarter, two dimes, and a nickel.
- Drawing a Y or a letter that follows L in the alphabet when a letter is drawn from the word PROBABILITY.

## EXAMPLE 1

A standard deck of 52 cards is shuffled, and one card is drawn at random. Find the probability that the card is:

**a.** a king or an ace **b.** red or an ace

***Solution*** **a.** There are four kings in the deck, so $P(\text{king}) = \frac{4}{52}$.

There are four aces in the deck, so $P(\text{ace}) = \frac{4}{52}$.

These are mutually exclusive events. The set of kings and the set of aces are disjoint sets, having no elements in common.

$$P(\text{king or ace}) = P(\text{king}) + P(\text{ace}) = \frac{4}{52} + \frac{4}{52} = \frac{8}{52} \text{ or } \frac{2}{13} \quad \textit{Answer}$$

**b.** There are 26 red cards in the deck, so $P(\text{red}) = \frac{26}{52}$.

There are four aces in the deck, so $P(\text{ace}) = \frac{4}{52}$.

There are two red aces in the deck, so $P(\text{red and ace}) = \frac{2}{52}$.

These are not mutually exclusive events. Two cards are both red and ace.

$$P(\text{red or ace}) = P(\text{red}) + P(\text{ace}) - P(\text{red and ace})$$
$$= \frac{26}{52} + \frac{4}{52} - \frac{2}{52} = \frac{28}{52} \text{ or } \frac{7}{13} \quad \textit{Answer}$$

***Alternative Solution*** There are 26 red cards and two more aces not already counted (the ace of spades, the ace of clubs). Therefore, there are 26 + 2, or 28, cards in this event. Then:

$$P(\text{red or ace}) = \frac{28}{52} \text{ or } \frac{7}{13} \quad \textit{Answer}$$

## EXAMPLE 2

There are two events, $A$ and $B$. Given that $P(A) = .3$, $P(B) = .5$, and $P(A \cap B) = .1$, find $P(A \cup B)$.

***Solution*** Since the probability of $A \cap B$ is not 0, $A$ and $B$ are not mutually exclusive.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= .3 + .5 - .1 = .7$$

*Answer* $P(A \cup B) = .7$

## EXAMPLE 3

A town has two newspapers, the *Times* and the *Chronicle*. One out of every two persons in the town subscribes to the *Times*, three out of every five persons in the town subscribe to the *Chronicle*, and three out of every ten persons in the town subscribe to both papers. What is the probability that a person in the town chosen at random subscribes to the *Times* or the *Chronicle*?

***Solution*** Subscribing to the *Times* and subscribing to the *Chronicle* are not mutually exclusive events.

$$P(\textit{Times}) = \tfrac{1}{2} \qquad P(\textit{Chronicle}) = \tfrac{3}{5} \qquad P(\textit{Times} \text{ and } \textit{Chronicle}) = \tfrac{3}{10}$$

$$\begin{aligned} P(\textit{Times} \text{ or } \textit{Chronicle}) &= P(\textit{Times}) + P(\textit{Chronicle}) - P(\textit{Times} \text{ and } \textit{Chronicle}) \\ &= \tfrac{1}{2} + \tfrac{3}{5} - \tfrac{3}{10} \\ &= \tfrac{5}{10} + \tfrac{6}{10} - \tfrac{3}{10} = \tfrac{8}{10} \text{ or } \tfrac{4}{5} \end{aligned}$$

*Answer*

## EXERCISES

### Writing About Mathematics

**1.** Let $A$ and $B$ be two events. Is it possible for $P(A \text{ or } B)$ to be less that $P(A)$? Explain why or why not.

**2.** If $B$ is a subset of $A$, which of the following is true: $P(A \text{ or } B) < P(A)$, $P(A \text{ or } B) = P(A)$, $P(A \text{ or } B) > P(A)$? Explain your answer.

### Developing Skills

**3.** A spinner consists of five equal sectors of a circle. The sectors are numbered 1 through 5, and when an arrow is spun, it is equally likely to stop on any sector. For a single spin of the arrow, determine whether or not the events are mutually exclusive and find the probability that the number on the sector is:

**a.** 3 or 4 **b.** odd or 2 **c.** 4 or less **d.** 2 or 3 or 4 **e.** odd or 3

**4.** A fair die is rolled once. Determine whether or not the events are mutually exclusive and find the probability that the number rolled is:

**a.** 3 or 4
**b.** odd or 2
**c.** 4 or less than 4
**d.** 2, 3, or 4
**e.** odd or 3
**f.** less than 2 or more than 5
**g.** less than 5 or more than 2
**h.** even or more than 3

**5.** From a standard deck of cards, one card is drawn. Determine whether or not the events are mutually exclusive and find the probability that the card will be:

**a.** a queen or an ace
**b.** a queen or a 7
**c.** a heart or a spade
**d.** a queen or a spade
**e.** a queen or a red card
**f.** a jack or a queen or a king
**g.** a 7 or a diamond
**h.** a club or a red card
**i.** an ace or a picture card

**6.** A bank contains two quarters, six dimes, three nickels, and five pennies. A coin is drawn at random. Determine whether or not the events are mutually exclusive and find the probability that the coin is:

**a.** a quarter or a dime
**b.** a dime or a nickel
**c.** worth 10 cents or more than 10 cents
**d.** worth 10 cents or less
**e.** worth 1 cent or more
**f.** a quarter, a nickel, or a penny

In 7–12, in each case choose the numeral preceding the expression that best completes the statement or answers the question.

**7.** If a single card is drawn from a standard deck, what is the probability that it is a 4 or a 9?

(1) $\frac{2}{52}$ (2) $\frac{8}{52}$ (3) $\frac{13}{52}$ (4) $\frac{26}{52}$

**8.** If a single card is drawn from a standard deck, what is the probability that it is a 4 or a diamond?

(1) $\frac{8}{52}$ (2) $\frac{16}{52}$ (3) $\frac{17}{52}$ (4) $\frac{26}{52}$

**9.** If $P(A) = .2$, $P(B) = .5$, and $P(A \cap B) = .1$, then $P(A \cup B) =$

(1) .6 (2) .7 (3) .8 (4) .9

**10.** If $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{2}$, and $P(A \text{ and } B) = \frac{1}{6}$, then $P(A \text{ or } B) =$

(1) $\frac{2}{5}$ (2) $\frac{2}{3}$ (3) $\frac{5}{6}$ (4) 1

**11.** If $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{2}$, and $P(A \cap B) = \frac{1}{8}$, then $P(A \cup B) =$

(1) $\frac{1}{8}$ (2) $\frac{5}{8}$ (3) $\frac{3}{4}$ (4) $\frac{7}{8}$

**12.** If $P(A) = .30$, $P(B) = .35$, and $(A \cap B) = \varnothing$, then $P(A \cup B) =$

(1) .05 (2) .38 (3) .65 (4) 0

## Applying Skills

**13.** Linda and Aaron recorded a CD together. Each sang some solos, and both of them sang some duets. Aaron recorded twice as many duets as solos, and Linda recorded six more solos than duets. If a CD player selects one of these songs at random, the probability that it will select a duet is $\frac{1}{4}$. Find the number of:

**a.** solos by Aaron
**b.** solos by Linda
**c.** duets

**14.** In a sophomore class of 340 students, some students study Spanish, some study French, some study both languages, and some study neither language. If $P(\text{Spanish}) = .70$, $P(\text{French}) = .40$, and $P(\text{Spanish } \textit{and} \text{ French}) = .25$, find:

**a.** the probability that a sophomore studies Spanish *or* French
**b.** the number of sophomores who study one or more of these languages

**15.** The Greenspace Company offers lawn care services and snow plowing in the appropriate seasons. Of the 600 property owners in town, 120 have contracts for lawn care, 90 for snow plowing, and 60 for both with the Greenspace Company. A new landscape company, the Earthpro Company offers the same services and begins a telephone campaign to attract customers, choosing telephone numbers of property owners at random. What is the probability that the Earthpro Company reaches someone who has a contract for lawn care or snow plowing with the Greenspace Company?

## 15-6 THE PROBABILITY OF (NOT *A*)

In rolling a fair die, we know that $P(4) = \frac{1}{6}$ since there is only one outcome for the event (rolling a 4). We can also say that $P(\textit{not}\ 4) = \frac{5}{6}$ since there are five outcomes for the event (not rolling a 4): 1, 2, 3, 5, and 6.

We can think of these probabilities in another way. The event (4) and the event (not 4) are mutually exclusive events. Also, the event (4 or not 4) is a certainty whose probability is 1.

$$\begin{aligned} P(4 \text{ or not } 4) &= P(4) + P(\text{not } 4) \\ 1 &= P(4) + P(\text{not } 4) \\ 1 - P(4) &= P(\text{not } 4) \\ 1 - \tfrac{1}{6} &= P(\text{not } 4) \\ \tfrac{5}{6} &= P(\text{not } 4) \end{aligned}$$

The event (not $A$) is the **complement** of event $A$, when the universal set is the sample space, $S$.

In general, if $P(A)$ is the probability that some given result will occur, and $P(\text{not } A)$ is the probability that the given result will not occur, then:

1. $P(A) + P(\text{not } A) = 1$
2. $P(A) = 1 - P(\text{not } A)$
3. $P(\text{not } A) = 1 - P(A)$

### Probability as a Sum

When sets are disjoint, we have seen that the probability of the union can be found by the rule $P(A \cup B) = P(A) + P(B)$. Since the possible outcomes that are singletons represent disjoint sets, we can say:

► **The probability of any event is equal to the sum of the probabilities of the singleton outcomes in the event.**

For example, when we draw a card from a standard deck, there are 52 singleton outcomes, each with a probability of $\frac{1}{52}$. Since all singleton events are disjoint, we can say:

$$\begin{aligned} P(\text{king}) &= P(\text{king of hearts}) + P(\text{king of diamonds}) + P(\text{king of spades}) \\ &\quad + P(\text{king of clubs}) \\ &= \tfrac{1}{52} + \tfrac{1}{52} + \tfrac{1}{52} + \tfrac{1}{52} \\ P(\text{king}) &= \tfrac{4}{52} \text{ or } \tfrac{1}{13} \end{aligned}$$

We also say:

► **The sum of the probabilities of all possible singleton outcomes for any sample space must always equal 1.**

For example, in tossing a coin,

$$P(S) = P(\text{heads}) + P\,(\text{tails}) = \tfrac{1}{2} + \tfrac{1}{2} = 1.$$

Also, in rolling a die,

$$\begin{aligned} P(S) &= P(1) + P(2) + P(3) + P(4) + P(5) + P(6) \\ &= \tfrac{1}{6} + \tfrac{1}{6} + \tfrac{1}{6} + \tfrac{1}{6} + \tfrac{1}{6} + \tfrac{1}{6} \\ &= 1 \end{aligned}$$

## EXAMPLE 1

Dr. Van Brunt estimates that 4 out every 10 patients that he will see this week will need a flu shot. What is the probability that the next patient he sees will *not* need a flu shot?

***Solution*** The probability that a patient will need a flu shot is $\frac{4}{10}$ or $\frac{2}{5}$.

The probability that a patient will not need a flu shot is $1 - \frac{2}{5} = \frac{3}{5}$. *Answer*

## EXAMPLE 2

A letter is drawn at random from the word ERROR.

**a.** Find the probability of drawing each of the different letters in the word.

**b.** Demonstrate that the sum of these probabilities is 1.

***Solution*** **a.** $P(E) = \frac{1}{5}$; $P(\text{R}) = \frac{3}{5}$; $P(\text{O}) = \frac{1}{5}$ *Answer*

**b.** $P(\text{E}) + P(\text{R}) + P(\text{O}) = \frac{1}{5} + \frac{3}{5} + \frac{1}{5} = \frac{5}{5} = 1$ *Answer*

## EXERCISES

### Writing About Mathematics

**1.** If event $A$ is a certainty, $P(A) = 1$. What must be true about $P(\text{not } A)$? Explain your answer.

**2.** If $A$ and $B$ are disjoint sets, what is $P(\text{not } A \text{ or not } B)$? Explain your answer.

### Developing Skills

**3.** A fair die is rolled once. Find each probability:

**a.** $P(3)$ **b.** $P(\text{not } 3)$ **c.** $P(\text{even})$

**d.** $P(\text{not even})$ **e.** $P(\text{less than } 3)$ **f.** $P(\text{not less than } 3)$

**g.** $P(\text{odd or even})$ **h.** $P[\text{not (odd or even)}]$ **i.** $P[\text{not } (2 \text{ or } 3)]$

**4.** From a standard deck of cards, one card is drawn. Find the probability that the card is:

**a.** a heart **b.** not a heart **c.** a picture card

**d.** not a picture card **e.** not an 8 **f.** not a red 6

**g.** not the queen of spades **h.** not an 8 or a 6

**5.** One letter is selected at random from the word PICNICKING.

**a.** Find the probability of drawing each of the different letters in the word.

**b.** Demonstrate that the sum of these probabilities is 1.

**6.** If the probability of an event happening is $\frac{1}{7}$, what is the probability of that event not happening?

**7.** If the probability of an event happening is .093, what is the probability of that event not happening?

**8.** A jar contains seven marbles, all the same size. Three are red and four are green. If a marble is chosen at random, find each probability:

**a.** $P(\text{red})$ **b.** $P(\text{green})$ **c.** $P(\text{not red})$

**d.** $P(\text{red or green})$ **e.** $P(\text{red and green})$ **f.** $P[\text{not (red or green)}]$

**9.** A box contains 3 times as many black marbles as green marbles, all the same size. If a marble is drawn at random, find the probability that it is:

**a.** black **b.** green **c.** not black **d.** black or green **e.** not green

**10.** A letter is chosen at random from the word PROBABILITY. Find each probability:

**a.** $P(\text{A})$ **b.** $P(\text{B})$ **c.** $P(\text{C})$

**d.** $P(\text{A or B})$ **e.** $P(\text{A or I})$ **f.** $P(\text{a vowel})$

**g.** $P(\text{not a vowel})$ **h.** $P(\text{A or B or L})$ **i.** $P(\text{A or not A})$

**11.** A single card is drawn at random from a well-shuffled deck of 52 cards. Find the probability that the card is:

**a.** a 6 **b.** a club **c.** the 6 of clubs

**d.** a 6 or a club **e.** not a club **f.** not a 6

**g.** a 6 or a 7 **h.** not the 6 of clubs **i.** a 6 and a 7

**j.** a black 6 **k.** a 6 or a black card **l.** a black card or not a 6

## Applying Skills

**12.** A bank contains three quarters, four dimes, and five nickels. A coin is drawn at random.

**a.** Find the probability of drawing:

(1) a quarter (2) a dime (3) a nickel

**b.** Demonstrate that the sum of the three probabilities given as answers in part **a** is 1.

**c.** Find the probability of not drawing:

(1) a quarter (2) a dime (3) a nickel

**13.** The weather bureau predicted a 30% chance of rain. Express *in fractional form*:

**a.** the probability that it will rain **b.** the probability that it will not rain

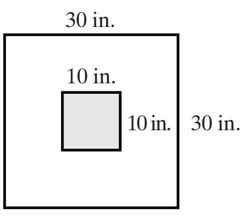
**14.** Mr. Jacobsen's mail contains two letters, three bills, and five ads. He picks up the first piece of mail without looking at it. Express, *in decimal form*, the probability that this piece of mail is:

**a.** a letter **b.** a bill **c.** an ad

**d.** a letter or an ad **e.** a bill or an ad **f.** not a bill

**g.** not an ad **h.** a bill and an ad

**15.** The square dartboard shown at the right, whose side measures 30 inches, has at its center a shaded square region whose side measures 10 inches. If darts directed at the board are equally likely to land anywhere on the board, what is the probability that a dart does *not* land in the shaded region?




**16.** A telephone keypad contains the ten digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Mabel is calling a friend. Find the probability that the last digit in her friend's telephone number is:

**a.** 6

**b.** 6 or a larger number

**c.** a number smaller than 6

**d.** 6 or an odd number

**e.** 6 or a number smaller than 6

**f.** not 6

**g.** 6 and an odd number

**h.** a number not larger than 6

**i.** a number smaller than 2 and larger than 6

**j.** a number smaller than 2 or larger than 6

**k.** a number smaller than 6 and larger than 2

**l.** a number smaller than 6 or larger than 2

## 15-7 THE COUNTING PRINCIPLE, SAMPLE SPACES, AND PROBABILITY

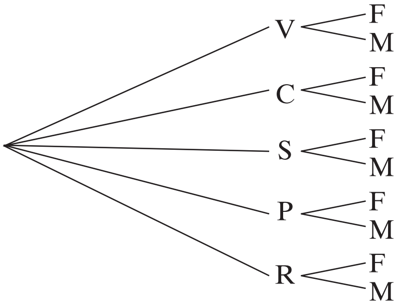
So far, we have looked at simple problems involving a single activity, such as rolling one die or choosing one card. More realistic problems occur when there are two or more activities, such as rolling two dice or dealing a hand of five cards. An event consisting of two or more activities is called a **compound event**.

Before studying the probability of events based on two or more activities, let us study ways to count the number of elements or outcomes in a sample space for two or more activities. For example:

A store offers five flavors of ice cream: vanilla, chocolate, strawberry, peach, and raspberry. A sundae can be made with either a hot fudge topping or a marshmallow topping. If a sundae consists of one flavor of ice cream and one topping, how many different sundaes are possible?

We will use letters to represent the five flavors of ice cream (V, C, S, P, R) and the two toppings (F, M). We can show the number of elements in the sample space in three ways:

1. **Tree Diagram.** The tree diagram at the right first branches out to show five flavors of ice cream. For each of these flavors the tree again branches out to show the two toppings. In all, there are 10 paths or branches to follow, each with one flavor of ice cream and one topping. These 10 branches show that the sample space consists of 10 possible outcomes, in this case, sundaes.



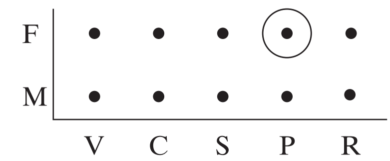
Tree diagram

2. **List of Ordered Pairs.** It is usual to order a sundae by telling the clerk the flavor of ice cream first and then the type of topping. This suggests a listing of ordered pairs. The first component of the ordered pair is the ice-cream flavor, and the second component is the type of topping. The set of pairs (ice cream, topping) is shown below.

   {(V, F), (C, F), (S, F), (P, F), (R, F), (V, M), (C, M), (S, M), (P, M), (R, M)}

   These 10 ordered pairs show that the sample space consists of 10 possible sundaes.

**3. Graph of Ordered Pairs.** Instead of listing pairs, we may construct a graph of the ordered pairs. At the right, the five flavors of ice cream appear on the horizontal scale or line, and the two toppings are on the vertical line. Each point in the graph represents an ordered pair. For example, the point circled shows the ordered pair (P, F), that is, (peach ice cream, fudge topping).



Graph of ordered pairs

This graph of 10 points, or 10 ordered pairs, shows that the sample space consists of 10 possible sundaes.

Whether we use a tree diagram, a list of ordered pairs, or a graph of ordered pairs, we recognize that the sample space consists of 10 sundaes. The number of elements in the sample space can be found by multiplication:

$$\underbrace{\text{number of flavors of ice cream}}_{5} \times \underbrace{\text{number of toppings}}_{2} = \underbrace{\text{number of possible sundaes}}_{10}$$

Suppose the store offered 30 flavors of ice cream and seven possible toppings. To find the number of elements in the sample space, we multiply:

$$30 \times 7 = 210 \text{ possible sundaes}$$

This simple multiplication procedure is known as the **counting principle**, because it enables us to count the number of elements in a sample space.

► ***The Counting Principle:*** **If one activity can occur in any of *m* ways and, following this, a second activity can occur in any of *n* ways, then both activities can occur in the order given in $m \times n$ ways.**

We can extend this rule to include three or more activities by extending the multiplication process. We can also display three or more activities by extending the branches on a tree diagram, or by listing ordered elements such as ordered triples and ordered quadruples.

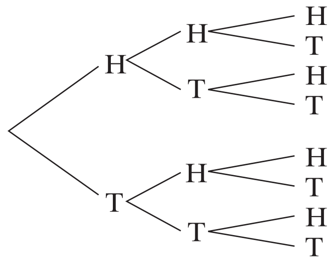
For example, a coin is tossed three times in succession.

- On the first toss, the coin may fall in either of two ways: a head or a tail.
- On the second toss, the coin may also fall in either of two ways.
- On the third toss, the coin may still fall in either of two ways.

By the counting principle, the sample space contains $2 \times 2 \times 2$ or 8 possible outcomes. By letting H represent a head and T represent a tail, we can illustrate the sample space by a tree diagram or by a list of ordered triples:

We did *not* attempt to draw a graph of this sample space because we would need a horizontal scale, a vertical scale, and a third scale, making the graph

Three Tosses of a Coin




(H, H, H)
(H, H, T)
(H, T, H)
(H, T, T)

(T, H, H)
(T, H, T)
(T, T, H)
(T, T, T)

Tree diagram

List of ordered triples

three-dimensional. Although such a graph can be drawn, it is too difficult at this time. We can conclude that:

1. Tree diagrams, or lists of ordered elements, are effective ways to indicate any compound event of two or more activities.
2. Graphs should be limited to ordered pairs, or to events consisting of exactly two activities.

**EXAMPLE 1**

The school cafeteria offers four types of salads, three types of beverages, and five types of desserts. If a lunch consists of one salad, one beverage, and one dessert, how many possible lunches can be chosen?

***Solution*** By the counting principle, we multiply the number of possibilities for each choice:

$$4 \times 3 \times 5 = 12 \times 5 = 60 \text{ possible lunches } \textit{Answer}$$

## Independent Events

The probability of rolling 5 on one toss of a die is $\frac{1}{6}$. What is the probability of rolling a pair of 5's when two dice are tossed?

When we roll two dice, the number obtained on the second die is in no way determined by of the result obtained on the first die. When we toss two coins, the face that shows on the second coin is in no way determined by the face that shows on the first coin.

When the result of one activity in no way influences the result of a second activity, the results of these activities are called **independent events**. In cases where two events are independent, we may extend the counting principle to find the probability that both independent events occur at the same time.

For instance, what is the probability that, when two dice are thrown, a 5 will appear on each of the dice? Let $S$ represent the sample space and $F$ represent the event (5 on both dice).

(1) Use the counting principle to find the number of elements in the sample space. There are 6 ways in which the first die can land and 6 ways in which the second die can land. Therefore, there are $6 \times 6$ or 36 pairs of numbers in the sample space, that is, $n(S) = 36$.

(2) There is only one face on each die that has a 5. Therefore there is $1 \times 1$ or 1 pair in the event $F$, that is $n(F) = 1$.

(3) $P(5 \text{ on both dice}) = \frac{n(F)}{n(S)} = \frac{1}{36}$

The probability of 5 on both dice can also be determined by using the probability of each of the independent events.

$$P(5 \text{ on first die}) = \tfrac{1}{6}$$
$$P(5 \text{ on second die}) = \tfrac{1}{6}$$
$$P(5 \text{ on both dice}) = P(5 \text{ on first}) \times P(5 \text{ on second}) = \tfrac{1}{6} \times \tfrac{1}{6} = \tfrac{1}{36}$$

We can extend the counting principle to help us find the probability of any two or more independent events.

► ***The Counting Principle for Probability:* $E$ and $F$ are independent events. The probability of event $E$ is $m$ $(0 \leq m \leq 1)$ and the probability of event $F$ is $n$ $(0 \leq n \leq 1)$. The probability of the event in which $E$ and $F$ occur jointly is the product $m \times n$.**

**Note 1:** The product $m \times n$ is within the range of values for a probability, namely, $0 \leq m \times n \leq 1$.

**Note 2:** Not all events are independent, and this simple product rule cannot be used to find the probability when events are not independent.

**EXAMPLE 2**

Mr. Gillen may take any of three buses, $A$ or $B$ or $C$, to get to the train station. He may then take the 6th Avenue train or the 8th Avenue train to get to work. The buses and trains arrive at random and are equally likely to arrive. What is the probability that Mr. Gillen takes the $B$ bus and the 6th Avenue train to get to work?

***Solution***

$$P(\text{B bus}) = \tfrac{1}{3} \text{ and } P(\text{6th Ave. train}) = \tfrac{1}{2}$$

Since the train taken is independent of the bus taken:

$$P(B \text{ bus and 6th Ave. train}) = P(B \text{ bus}) \times P(\text{6th Ave. train})$$
$$= \tfrac{1}{3} \times \tfrac{1}{2} = \tfrac{1}{6} \text{ \textit{Answer}}$$

## EXERCISES

### Writing About Mathematics

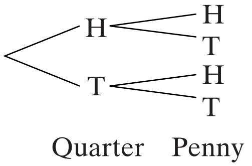
**1.** Judy said that if a quarter and a nickel are tossed, there are three equally likely outcomes; two heads, two tails, or one head and one tail. Do you agree with Judy? Explain why or why not.

**2. a.** When a green die and a red die are rolled, is the probability of getting a 2 on the green die and a 3 on the red die the same as the probability of getting 3 on both dice? Explain why or why not.

**b.** When rolling two fair dice, is the probability of getting a 2 and a 3 the same as the probability of getting two 3's? Explain why or why not.

### Developing Skills

**3.** A quarter and a penny are tossed simultaneously. Each coin may fall heads or tails. The tree diagram at the right shows the sample space involved.




**a.** List the sample space as a set of ordered pairs.

**b.** Use the counting principle to demonstrate that there are four outcomes in the sample space.

**c.** In how many outcomes do the coins both fall heads up?

**d.** In how many outcomes do the coins land showing one head and one tail?

**4.** Two dice are rolled simultaneously. Each die may land with any one of the six numbers faceup.

**a.** Use the counting principle to determine the number of outcomes in this sample space.

**b.** Display the sample space by drawing a graph of the set of ordered pairs.

**5.** A sack contains four marbles: one red, one blue, one white, one green. One marble is drawn and placed on the table. Then a second marble is drawn and placed to the right of the first.

**a.** How many possible marbles can be selected on the first draw?

**b.** How many possible marbles can be selected on the second draw?

**c.** How many possible ordered pairs of marbles can be drawn?

**d.** Draw a tree diagram to show all of the possible outcomes for this experiment. Represent the marbles by $R$, $B$, $W$, and $G$.

**6.** A sack contains four marbles: one red, one blue, one white, one green. One marble is drawn, its color noted and then it is replaced in the sack. A second marble is drawn and its color noted to the right of the first.

**a.** How many possible colors can be noted on the first draw?

**b.** How many possible colors can be noted on the second draw?

**c.** How many possible ordered pairs of colors can be noted?

**d.** Draw a tree diagram to show all of the possible outcomes for this experiment. Represent the marbles by $R$, $B$, $W$, and $G$.

**7.** A fair coin and a six-sided die are tossed simultaneously. What is the probability of obtaining:

**a.** a head on the coin?

**b.** a 4 on the die?

**c.** a head on the coin and a 4 on the die on a single throw?

**8.** A fair coin and a six-sided die are tossed simultaneously. What is the probability of obtaining on a single throw:

**a.** a head and a 3?

**b.** a head and an even number?

**c.** a tail and a number less than 5?

**d.** a tail and a number greater than 4?

**9.** Two fair coins are tossed. What is the probability that both land heads up?

**10.** Three fair coins are tossed. **a.** Find $P(\text{H, H, H})$. **b.** Find $P(\text{T, T, T})$.

**11.** A fair spinner contains congruent sectors, numbered 1 through 8. If the arrow is spun twice, find the probability that it lands:

**a.** (7, 7) **b.** not (7, 7) **c.** (not 7, not 7) **d.** (7, not 7) or (not 7, 7)

**e.** (2, 8) **f.** not (2, 8) **g.** (not 2, not 8) **h.** (2, not 8) or (not 2, 8)

**12.** A fair coin is tossed 50 times and lands heads up each time. What is the probability that it will land heads up on the next toss? Explain your answer.

## Applying Skills

**13.** Tell how many possible outfits consisting of one shirt and one pair of pants Terry can choose if he owns:

**a.** 5 shirts and 2 pairs of pants

**b.** 10 shirts and 4 pairs of pants

**c.** 6 shirts and 6 pairs of pants

**14.** There are 10 doors into the school and eight staircases from the first floor to the second. How many possible ways are there for a student to go from outside the school to a classroom on the second floor?

**15.** A tennis club has 15 members: eight women and seven men. How many different teams may be formed consisting of one woman and one man on the team?

**16.** A dinner menu lists two soups, seven main courses, and three desserts. How many different meals consisting of one soup, one main course, and one dessert are possible?

**17.** The school cafeteria offers the menu shown.

| Main Course | Dessert | Beverage |
|---|---|---|
| Pizza | Yogurt | Milk |
| Frankfurter | Fruit salad | Juice |
| Ham sandwich | Jello | |
| Tuna sandwich | Apple pie | |
| Veggie burger | | |

**a.** How many meals consisting of one main course, one dessert, and one beverage can be selected from this menu?

**b.** Joe does not like ham and tuna. How many meals (again, one main course, one dessert, and one beverage) can Joe select, not having ham and not having tuna?

**c.** If the pizza, frankfurters, yogurt, and fruit salad have been sold out, how many different meals can JoAnn select from the remaining menu?

**18.** A teacher gives a quiz consisting of three questions. Each question has as its answer either true (T) or false (F).

**a.** Using T and F, draw a tree diagram to show all possible ways the questions can be answered.

**b.** List this sample space as a set of ordered triples.

**19.** A test consists of multiple-choice questions. Each question has four choices. Tell how many possible ways there are to answer the questions on the test if the test consists of the following number of questions:

**a.** 1 question **b.** 3 questions **c.** 5 questions **d.** $n$ questions

**20.** Options on a bicycle include two types of handlebars, two types of seats, and a choice of 15 colors. The bike may also be ordered in ten-speed, in three-speed, or standard. How many possible versions of a bicycle can a customer choose from, if he selects a specific type of handlebars, type of seat, color, and speed?

**21.** A state issues license plates consisting of letters and numbers. There are 26 letters, and the letters may be repeated on a plate; there are 10 digits, and the digits may be repeated. Tell how many possible license plates the state may issue when a license consists of each of the following:

**a.** 2 letters, followed by 3 numbers

**b.** 2 numbers, followed by 3 letters

**c.** 4 numbers, followed by 2 letters

**22.** In a school cafeteria, the menu rotates so that $P(\text{hamburger}) = \frac{1}{4}$, $P(\text{apple pie}) = \frac{2}{3}$, and $P(\text{soup}) = \frac{4}{5}$. The selection of menu items is random so that the appearance of hamburgers, apple pie, and soup are independent events. On any given day, what is the probability that the cafeteria offers hamburger, apple pie, and soup on the same menu?

**23.** A quiz consists of true-false questions only. Harry has not studied, and he guesses every answer. Find the probability that he will guess correctly to get a perfect score if the test consists of: **a.** 1 question **b.** 4 questions **c.** $n$ questions

**24.** The probability of the Tigers beating the Cougars is $\frac{2}{3}$. The probability of the Tigers beating the Mustangs is $\frac{1}{4}$. If the Tigers play one game with the Cougars and one game with the Mustangs, find the probability that the Tigers: **a.** win both games **b.** lose both games

**25.** On Main Street in Pittsford, there are two intersections that have traffic lights. The lights are not timed to accommodate traffic. They are independent of one another. At each of the intersections, $P(\text{red light}) = .3$ and $P(\text{green light}) = .7$ for cars traveling along Main Street. Find the probability that a car traveling on Main Street will be faced with each set of given conditions at the two traffic lights shown.

**a.** Both lights are red.

**b.** Both lights are green.

**c.** The first light is red, and the second is green.

**d.** The first light is green, and the second is red.

**e.** At least one light is red, that is, not both lights are green.

**f.** Both lights are the same color.

**26.** A manufacturer of radios knows that the probability of a defect in any of his products is $\frac{1}{400}$. If 10,000 radios are manufactured in January, how many are likely to be defective?

**27.** Past records from the weather bureau indicate that the probability of rain in August on Cape Cod is $\frac{2}{7}$. If Joan goes to Cape Cod for 2 weeks in August, how many days will it probably rain if the records hold true?

**28.** A nationwide fast-food chain has a promotion, distributing to customers 2,000,000 coupons for the prizes shown below. Each coupon awards the customer one of the following prizes.

1 Grand Prize: $25,000 cash
2 Second-Place Prizes: New car
100 Third-Place Prizes: New TV
Fourth-Place Prizes: Free meal
Consolation Prizes: 25 cents off any purchase

**a.** Find the probability of winning:

(1) the grand prize (2) a new car (3) a new TV

**b.** If a customer has one coupon, what is the probability of winning one of the first three prizes (cash, a car, or a TV)?

**c.** If the probability of winning a free meal is $\frac{1}{400}$, how many coupons are marked as fourth-place prizes?

**d.** How many coupons are marked "25 cents off any purchase"?

**e.** If a customer has one coupon, what is the probability of not winning one of the first three prizes?

# 15-8 PROBABILITIES WITH TWO OR MORE ACTIVITIES

## Without Replacement

Two cards are drawn at random from an ordinary pack of 52 cards. In this situation, a single card is drawn from a deck of 52 cards, and then a second card is drawn from the remaining 51 cards in the deck. What is the probability that both cards drawn are kings?

On the first draw, there are four kings in the deck of 52 cards, so

$$P(\text{first king}) = \frac{4}{52}$$

If a second card is drawn without replacing the first king selected, there are now only three kings in the 51 cards remaining. Therefore we are considering the probability of drawing a king, given that a king has already been drawn.

$$P(\text{second king}) = \frac{3}{51}$$

By the counting principle for probabilities:

$$P(\text{both kings}) = P(\text{first king}) \times P(\text{second king}) = \frac{4}{52} \times \frac{3}{51} = \frac{1}{13} \times \frac{1}{17} = \frac{1}{221}$$

This result could also have been obtained using the counting principle. The number of elements in the sample space is $52 \times 51$. The number of elements in the event (two kings) is $4 \times 3$. Then,

$$P(\text{both kings}) = \frac{4 \times 3}{52 \times 51} = \frac{12}{2{,}652} = \frac{1}{221}$$

This is called a problem **without replacement** because the first king drawn was not placed back into the deck. These are **dependent events** because the probability of a king on the second draw depends on whether or not a king appeared on the first draw.

In general, if $A$ and $B$ are two dependent events:

$$\boldsymbol{P(A \text{ and } B) = P(A) \times P(B \text{ given that } A \text{ has occurred})}$$

Earlier in the chapter we discussed $P(A \text{ and } B)$ where $(A \text{ and } B)$ is a single event that satisfies both conditions. Here $(A \text{ and } B)$ denotes two dependent events with $A$ the outcome of one event and $B$ the outcome of the other. The conditions of the problem will indicate which of these situations exists. When $A$ and $B$ are dependent events, $P(A \text{ and } B)$ can also be written as $P(A, B)$.

## With Replacement

A card is drawn at random from an ordinary deck, the card is placed back into the deck, and a second card is then drawn and replaced. In this situation, it is clear the deck contains 52 cards each time that a card is drawn and that the same card could be drawn twice. What is the probability that the card drawn each time is a king?

On the first draw, there are four kings in the deck of 52 cards.

$$P(\text{first king}) = \frac{4}{52}$$

If the first king drawn is now placed back into the deck, then, on the second draw, there are again four kings in the deck of 52 cards.

$$P(\text{second king}) = \frac{4}{52}$$

By the counting principle for probabilities:

$$P(\text{both kings}) = P(\text{first king}) \times P(\text{second king}) = \frac{4}{52} \times \frac{4}{52} = \frac{1}{13} \times \frac{1}{13} = \frac{1}{169}$$

This is called a problem **with replacement** because the first card drawn was placed back into the deck. In this case, the events are independent because the probability of a king on the second draw does not depend on whether or not a king appeared on the first draw. Since the card drawn is replaced, the number of cards in the deck and the number of kings in the deck remain constant. In this case,

$$P(B \text{ given that } A \text{ has occurred}) = P(B).$$

In general, if $A$ and $B$ are two independent events,

$$\boldsymbol{P(A \text{ and } B) = P(A) \times P(B)}$$

Rolling two dice is similar to drawing two cards with replacement because the number of faces on each die remains constant, as did the number of cards in the deck. Typical problems with replacement include rolling dice, tossing coins (each coin always has two sides), and spinning arrows.

## KEEP IN MIND

1. If the problem does not specifically mention *with replacement* or *without replacement*, ask yourself: "Is this problem with or without replacement?" or "Are the events dependent or independent?"
2. For many compound events, the probability can be determined most easily by using the counting principle.
3. Every probability problem can always be solved by:
   - counting the number of elements in the sample space, $n(S)$;
   - counting the number of outcomes in the event, $n(E)$;
   - substituting these numbers in the probability formula,

$$P(E) = \frac{n(E)}{n(S)}$$

## Conditional Probability

The previous discussion involved the concept of **conditional probability**. For both dependent and independent events, in order to find the probability of $A$ followed by $B$, it is necessary to calculate the probability that $B$ occurs given that $A$ has occurred.

Notation for conditional probability is $P(B \text{ given that } A \text{ has occurred}) = P(B \mid A)$. Then the following statement is true for both dependent and independent events:

$$\boldsymbol{P(A \text{ and } B) = P(A) \times P(B \mid A)}$$

If $A$ and $B$ independent events, $P(B \mid A) = P(B)$. Therefore, for independent events:

$$\boldsymbol{P(A \text{ and } B) = P(A) \times P(B \mid A) = P(A) \times P(B)}$$

The general formula $P(A \text{ and } B) = P(A) \times P(B \mid A)$ can be solved for $P(B \mid A)$:

$$\boldsymbol{P(B \mid A) = \frac{P(A \text{ and } B)}{P(A)}}$$

For example, suppose a box contains one red marble, one blue marble, one green marble, and one yellow marble. Two marbles are drawn without replacement. Let $R$ be the event {red marble} and $Y$ be the event {yellow marble}. The probability of $R$ is $\frac{1}{4}$ and the probability of $Y$ is $\frac{1}{4}$. If we want to find the probability of drawing a red marble *followed by* a yellow marble, $R$ and $Y$ are dependent events. We need the probability of $Y$ given that $R$ has occurred, which is $\frac{1}{3}$ since once the red marble has been drawn, only 3 marbles remain, one of which is yellow.

$$\begin{aligned} P(R \text{ followed by } Y) = P(R \text{ and } Y) &= P(R) \times P(Y \mid R) \\ &= \tfrac{1}{4} \times \tfrac{1}{3} \\ &= \tfrac{1}{12} \end{aligned}$$

This result can be shown by displaying the sample space.

$$\begin{matrix} (R, B) & (B, R) & (G, R) & (Y, R) \\ (R, G) & (B, G) & (G, B) & (Y, B) \\ (R, Y) & (B, Y) & (G, Y) & (Y, G) \end{matrix}$$

There are 12 possible outcomes in the sample space and one of them is $(R, Y)$. Therefore,

$$P(R, Y) = \tfrac{1}{12}.$$

## EXAMPLE 1

Three fair dice are thrown. What is the probability that all three dice show a 5?

***Solution*** These are independent events. There are six possible faces that can come up on each die.

On the first die, there is one way to obtain a 5 so $P(5) = \frac{1}{6}$.

On the second die, there is one way to obtain a 5 so $P(5) = \frac{1}{6}$.

On the third die, there is one way to obtain a 5 so $P(5) = \frac{1}{6}$.

By the counting principle:

$$\begin{aligned} P(5 \text{ on each die}) &= P(5 \text{ on first}) \times P(5 \text{ on second}) \times P(5 \text{ on third}) \\ &= \tfrac{1}{6} \times \tfrac{1}{6} \times \tfrac{1}{6} \\ &= \tfrac{1}{216} \text{ Answer} \end{aligned}$$

## EXAMPLE 2

If two cards are drawn from an ordinary deck without replacement, what is the probability that the cards form a pair (two cards of the same face value but different suits)?

***Solution*** These are dependent events. On the first draw, any card at all may be chosen, so:

$$P(\text{any card}) = \tfrac{52}{52}$$

There are now 51 cards left in the deck. Of these 51, there are three that match the first card taken, to form a pair, so:

$$P(\text{second card forms a pair}) = \tfrac{3}{51}$$

Then:

$$\begin{aligned} P(\text{pair}) &= P(\text{any card}) \times P(\text{second card forms a pair}) \\ &= \tfrac{52}{52} \times \tfrac{3}{51} \\ &= 1 \times \tfrac{1}{17} = \tfrac{1}{17} \text{ Answer} \end{aligned}$$

## EXAMPLE 3

A jar contains four white marbles and two blue marbles, all the same size. A marble is drawn at random and not replaced. A second marble is then drawn from the jar. Find the probability that:

**a.** both marbles are white **b.** both marbles are blue **c.** both marbles are the same color

***Solution*** These are dependent events.

**a.** On the first draw:

$$P(\text{white}) = \tfrac{4}{6}$$

Since the white marble drawn is not replaced, five marbles, of which three are white, are left in the jar. On the second draw:

$$P(\text{white given that the first was white}) = \tfrac{3}{5}$$

Then:

$$P(\text{both white}) = \tfrac{4}{6} \times \tfrac{3}{5} = \tfrac{12}{30} \text{ or } \tfrac{2}{5} \quad \textit{Answer}$$

**b.** Start with a full jar of six marbles of which two are blue.

On the first draw:

$$P(\text{blue}) = \tfrac{2}{6}$$

Since the blue marble drawn is not replaced, five marbles, of which only one is blue, are left in the jar. On the second draw:

$$P(\text{blue given that the first was blue}) = \tfrac{1}{5}$$

Then:

$$P(\text{both blue}) = \tfrac{2}{6} \times \tfrac{1}{5} = \tfrac{2}{30} \text{ or } \tfrac{1}{15} \quad \textit{Answer}$$

**c.** If both marbles are the same color, both are white or both are blue. These are disjoint or mutually exclusive events:

$$P(A \text{ or } B) = P(A) + P(B)$$

Therefore:

$$\begin{aligned} P(\text{both white or both blue}) &= P(\text{both white}) + P(\text{both blue}) \\ &= \tfrac{2}{5} + \tfrac{1}{15} \\ &= \tfrac{6}{15} + \tfrac{1}{15} \\ &= \tfrac{7}{15} \quad \textit{Answer} \end{aligned}$$

**Note:** By considering the complement of the set named in part **c**, we can easily determine the probability of drawing two marbles of *different* colors.

$$\begin{aligned} P[\text{not (both white or both blue)}] &= 1 - P(\text{both white or both blue}) \\ &= 1 - \tfrac{7}{15} \\ &= \tfrac{8}{15} \end{aligned}$$

## EXAMPLE 4

A fair die is rolled.

**a.** Find the probability that the die shows a 4 given that the die shows an even number.

**b.** Find the probability that the die shows a 1 given that the die shows a number less than 5.

***Solution*** **a.** Use the formula for conditional probability.

$$P(4 \text{ given an even number}) = P(4 \mid \text{even}) = \frac{P(4 \text{ and even})}{P(\text{even})}$$

The event "4 and even" occurs whenever the outcome is 4. Therefore, $P(4 \text{ and even}) = P(4)$.

$$P(4 \mid \text{even}) = \frac{P(4 \text{ and even})}{P(\text{even})} = \frac{P(4)}{P(\text{even})} = \frac{\frac{1}{6}}{\frac{3}{6}} = \frac{1}{3} \text{ Answer}$$

**b.** The event "1 and less than 5" occurs whenever the outcome is 1. Therefore, $P(1 \text{ and less than } 5) = P(1)$.

$$P(1 \mid \text{less than } 5) = \frac{P(1 \text{ and less than } 5)}{P(\text{less than } 5)} = \frac{P(1)}{P(\text{less than } 5)} = \frac{\frac{1}{6}}{\frac{5}{6}} = \frac{1}{5} \text{ Answer}$$

## EXAMPLE 5

Fred has two quarters and one nickel in his pocket. The pocket has a hole in it, and a coin drops out. Fred picks up the coin and puts it back into his pocket. A few minutes later, a coin drops out of his pocket again.

**a.** Draw a tree diagram or list the sample space for all possible pairs that are outcomes to describe the coins that fell.

**b.** What is the probability that the same coin fell out of Fred's pocket both times?

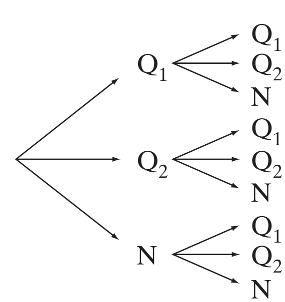
**a.** What is the probability that the two coins that fell have a total value of 30 cents?

**b.** What is the probability that a quarter fell out at least once?

***Solution*** **a.** Because there are two quarters, use subscripts.

The three coins are $\{Q_1, Q_2, N\}$, where Q represents a quarter and N represents a nickel. This is a problem with replacement. The events are independent.

$$\{(Q_1, Q_1), (Q_1, Q_2), (Q_1, N), (Q_2, Q_1), (Q_2, Q_2), (Q_2, N), (N, Q_1), (N, Q_2), (N, N)\}$$



**b.** Of the nine outcomes, three name the same coin both times: $(Q_1, Q_1)$, $(Q_2, Q_2)$ and (N, N). Therefore:

$$P(\text{same coin}) = \tfrac{3}{9} \text{ or } \tfrac{1}{3} \quad \textit{Answer}$$

**c.** Of the nine outcomes, the four that consist of a quarter and a nickel total 30 cents: $(Q_1, N)$, $(Q_2, N)$, $(N, Q_1)$, $(N, Q_2)$.

$$P(\text{total value of 30 cents}) = \tfrac{4}{9} \quad \textit{Answer}$$

**d.** Of the nine outcomes, only (N, N) does not include a quarter. Eight contain at least one quarter, that is, one or more quarters.

$$P(\text{at least one quarter}) = \tfrac{8}{9} \quad \textit{Answer}$$

**EXAMPLE 6**

Of Roosevelt High School's 1,000 students, 300 are athletes, 200 are in the Honor Roll, and 120 play sports and are in the Honor Roll. What is the probability that a randomly chosen student who plays a sport is also in the Honor Roll?

***Solution*** Let $A$ = the event that the student is an athlete, and

$B$ = the event that the student is in the Honor Roll.

Then, $A$ and $B$ = the event that a student is an athlete *and* is in the Honor Roll.

Therefore, the conditional probability that the student is in the Honor Roll given that he or she is an athlete is:

$$P(B \mid A) = \frac{P(A \text{ and } B)}{P(A)}$$

$$= \frac{\frac{120}{1{,}000}}{\frac{300}{1{,}000}} = \frac{120}{300} = .4 \quad \textit{Answer}$$

## EXERCISES

### Writing About Mathematics

**1.** The name of each person who attends a charity luncheon is placed in a box and names are drawn for first, second, and third prize. No one can win more than one prize. Is the probability of winning first prize greater than, equal to, or less than the probability of winning second prize? Explain your answer.

**2.** Six tiles numbered 1 through 6 are placed in a sack. Two tiles are drawn. Is the probability of drawing a pair of tiles whose numbers have a sum of 8 greater than, equal to, or less than the probability of obtaining a sum of 8 when rolling a pair of dice? Explain your answer.

## Developing Skills

3. A jar contains two red and five yellow marbles. If one marble is drawn at random, what is the probability that the marble drawn is: **a.** red? **b.** yellow?

4. A jar contains two red and five yellow marbles. A marble is drawn at random and then replaced. A second draw is made at random. Find the probability that:

   **a.** both marbles are red

   **b.** both marbles are yellow

   **c.** both marbles are the same color

   **d.** the marbles are different in color

5. A jar contains two red and five yellow marbles. A marble is drawn at random. Then without replacement, a second marble is drawn at random. Find the probability that:

   **a.** both marbles are red

   **b.** both marbles are yellow

   **c.** both marbles are the same color

   **d.** the marbles are different in color

   **e.** the second marble is red given that the first is yellow

6. In an experiment, an arrow is spun twice on a circular board containing four congruent sectors numbered 1 through 4. The arrow is equally likely to land on any one of the sectors.

   **a.** Indicate the sample space by drawing a tree diagram or writing a set of ordered pairs.

   **b.** Find the probability of spinning the digits 2 and 3 in that order.

   **c.** Find the probability that the same digit is spun both times.

   **d.** What is the probability that the first digit spun is larger than the second?

7. A jar contains nine orange disks and three blue disks. A girl chooses one at random and then, without replacing it, chooses another. Let O represent orange and B represent blue.

   **a.** Find the probability of each of the following outcomes:

   (1) (O, O) (2) (O, B) (3) (B, O) (4) (B, B)

   **b.** Now, find for the disks chosen, the probability that:

   (1) neither was orange

   (2) only one was blue

   (3) at least one was orange

   (4) they were the same color

   (5) at most one was orange

   (6) they were the same disk

   (7) the second is orange given that the first was blue

8. A card is drawn at random from a deck of 52 cards. Given that it is a red card, what is the probability that it is: **a.** a heart? **b.** a king?

9. A fair coin is tossed three times.

   **a.** What is the probability of getting all heads, given that the first toss is heads?

   **b.** What is the probability of getting all heads, given that the first two tosses are heads?

## Applying Skills

10. Sal has a bag of hard candies: three are lemon (L) and two are grape (G). He eats two of the candies while waiting for a bus, selecting them at random one after another.

**a.** Using subscripts, draw a tree diagram or list the sample space of all possible outcomes showing which candies are eaten.

**b.** Find the probability of each of the following outcomes:

(1) both candies are lemon
(2) neither candy is lemon
(3) the candies are the same flavor
(4) at least one candy is lemon

**c.** What is the probability that the second candy that Sal ate was lemon given that the first was grape?

**11.** Carol has five children: three girls and two boys. One of her children was late for lunch. Later that day, one of her children was late for dinner.

**a.** Indicate the sample space by a tree diagram or list of ordered pairs showing which children were late.

**b.** If each child was equally likely to be late, find the probability of each outcome below.

(1) Both children who were late were girls.
(2) Both children who were late were boys.
(3) The same child was late both times.
(4) At least one of the children who was late was a boy.

**c.** What is the probability that the child who was late for dinner was a boy given that the child who was late for lunch was a girl?

**12.** Several players start playing a game, each with a full deck of 52 cards. Each player draws two cards at random, one at a time without replacement. Find the probability that:

**a.** Flo draws two jacks
**b.** Frances draws two hearts
**c.** Jerry draws two red cards
**d.** Mary draws two picture cards
**e.** Ann does not draw a pair
**f.** Stephen draws two black kings
**g.** Carrie draws a 10 on her second draw given that the first was a 5
**h.** Bill draws a heart and a club in that order
**i.** Ann draws a king on her second draw given that the first was not a king

**13.** Saverio has four coins: a half dollar, a quarter, a dime, and a nickel. He chooses one of the coins at random and puts it in a bank. Later he chooses another coin and also puts that in the bank.

**a.** Indicate the sample space of coins saved.

**b.** If each coin is equally likely to be saved, find the probability that:

(1) the coins saved will be worth a total of 35 cents
(2) the coins saved will add to an even amount
(3) the coins saved will include the half-dollar
(4) the coins saved will be worth a total of less than 30 cents
(5) the second coin saved will be worth more than 20 cents given that the first coin saved was worth more than 20 cents.
(6) the second coin saved will be worth more than 20 cents given that the first coin saved was worth less than 20 cents.

**14.** Farmer Brown must wake up before sunrise to start his chores. Dressing in the dark, he reaches into a drawer and pulls out two loose socks. There are eight white socks and six red socks in the drawer.

**a.** Find the probability that both socks are:
(1) white (2) red (3) the same color (4) not the same color

**b.** Find the minimum number of socks Farmer Brown must pull out of the drawer to guarantee that he will get a matching pair.

**15.** Tillie is approaching the toll booth on an expressway. She has three quarters and four dimes in her purse. She takes out two coins at random from her purse. Find the probability of each outcome.

**a.** Both coins are quarters.

**b.** Both coins are dimes.

**c.** The coins are a dime and a quarter, in any order.

**d.** The value of the coins is enough to pay the 35-cent toll.

**e.** The value of the coins is not enough to pay the 35-cent toll given that one of the coins is a dime.

**f.** The value of the coins was 35 cents given that one coin was a quarter.

**g.** At least one of the coins picked was a quarter.

**16.** One hundred boys and one hundred girls were asked to name the current Secretary of State. Thirty boys and sixty girls knew the correct name. One of these boys and girls is selected at random.

**a.** What is the probability that the person selected knew the correct name?

**b.** What is the probability that the person selected is a girl, given that that person knew the correct name?

**c.** What is the probability that the person selected knew the correct name, given that the person is a boy?

**17.** In a graduating class of 400 seniors, 200 were male and 200 were female. The students were asked if they had ever downloaded music from an online music store. 75 of the male students and 70 of the female students said that they had downloaded music.

**a.** What is the probability that a randomly chosen senior has downloaded music?

**b.** What is the probability that a randomly chosen senior has downloaded music given that the senior is male?

**18.** Gracie baked one dozen sugar cookies and two dozen brownies. She then topped two-thirds of the cookies and half the brownies with chocolate frosting.

**a.** What is the probability that a randomly chosen treat is an unfrosted brownie?

**b.** What is the probability that a baked good chosen at random is a cookie given that it is frosted?

**19.** In a game of Tic-Tac-Toe, the first player can put an X in any of the four corner squares, four edge squares, or the center square of the grid. The second player can then put an O in any of the eight remaining open squares.

**a.** What is the probability that the second player will put an O in the center square given the first player has put an X in an edge square?

**b.** If the first player puts an X in a corner square, what is the probability that the second player will put an O in a corner square?

**c.** In a game of Tic-Tac-Toe, the first player puts an X in the center square and the second player puts an O in a corner square. What is the probability that the first player will put her next X in an edge square?

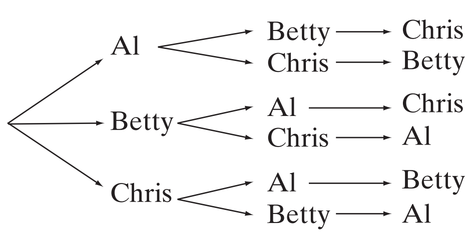
**20.** Of the 150 members of the high school marching band, 30 play the trumpet, 40 are in the jazz band, and 18 play the trumpet and are also in the jazz band.

**a.** What is the probability that a randomly chosen member of the marching band plays the trumpet but is not in the jazz band?

**b.** What is the probability that a randomly chosen member of the marching band is also in the jazz band but does not play the trumpet?

## 15-9 PERMUTATIONS

A teacher has announced that Al, Betty, and Chris, three students in her class, will each give an oral report today. How many possible ways are there for the teacher to choose the order in which these students will give their reports?




A tree diagram shows that there are six possible orders or arrangements. For example, Al, Betty, Chris is one possible arrangement; Al, Chris, Betty is another. Each of these arrangements is called a permutation. A **permutation** is an arrangement of objects or things in some specific order. (In discussing permutations, the words "objects" or "things" are used in a mathematical sense to include all elements in question, whether they are people, numbers, or inanimate objects.)

The six possible permutations in this case may also be shown as a set of ordered triples. Here, we let A represent Al, B represent Betty, and C represent Chris:

$$\{(A, B, C), (A, C, B), (B, A, C), (B, C, A), (C, A, B), (C, B, A)\}$$

Let us see, from another point of view, why there are six possible orders. We know that any one of three students can be called on to give the first report. Once the first report is given, the teacher may call on any one of the two remaining students. After the second report is given, the teacher must call on the one remaining student. Using the counting principle, we see that there are $3 \times 2 \times 1$ or 6 possible orders.

Consider another situation. A chef is preparing a recipe with 10 ingredients. He puts all of one ingredient in a bowl, followed by all of another ingredient, and so on. How many possible orders are there for placing the 10 ingredients in a bowl? Using the counting principle, we have:

$$10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 3{,}628{,}800 \text{ possible ways}$$

If there are more than 3 million possible ways of placing 10 ingredients in a bowl, can you imagine in how many ways 300 people who want to buy tickets for a football game can be arranged in a line? Using the counting principle, we find the number of possible orders to be $300 \times 299 \times 298 \times \cdots \times 3 \times 2 \times 1$. To symbolize such a product, we make use of the **factorial symbol**, **!**. We represent the product of these 300 numbers by the symbol 300!, read as "three hundred factorial" or "factorial 300."

## Factorials

In general, for any natural number $n$, we define ***n* factorial** or **factorial *n*** as follows:

**DEFINITION**

$$n! = n(n-1)(n-2)(n-3) \times \cdots \times 3 \times 2 \times 1$$

Note that 1! is the natural number 1.

Calculators can be used to evaluate factorials. On a graphing calculator, the factorial function is found by first pressing MATH and then using the left arrow key to highlight the PRB menu. For example, to evaluate 5!, use the following sequence of keys:

ENTER: 5 MATH ◀ 4 ENTER

DISPLAY:

```
5!
                 120
```

Of course, whether a calculator does or does not have a factorial function, we can always use repeated multiplication to evaluate a factorial:

$$5! = 5 \times 4 \times 3 \times 2 \times 1$$

When using a calculator, we must keep in mind that factorial numbers are usually very large. If the number of digits in a factorial exceeds the number of places in the display of the calculator, the calculator will shift from standard decimal notation to scientific notation. For example, evaluate 15! on a calculator:

ENTER: 15 MATH ◄ 4 ENTER

DISPLAY:

```
15!
      1.307674368E12
```

The number in the display can be written in scientific notation or in decimal notation.

$$1.307674368 \text{ E } 12 = 1.307674368 \times 10^{12} = 1{,}307{,}674{,}368{,}000$$

## Representing Permutations

We have said that permutations are arrangements of objects in different orders. For example, the number of different orders in which four people can board a bus is 4! or $4 \times 3 \times 2 \times 1$ or 24. There are 24 permutations, that is, 24 different orders or arrangements, of these four people, in which all four of them get on the bus.

We may also represent this number of permutations by the symbol ${}_4P_4$. The symbol ${}_4P_4$ is read as: "the number of permutations of four objects taken four at a time." Here, the letter $P$ represents the word *permutation*.

The small $_4$ written to the lower left of $P$ tells us that four objects are available to be used in an arrangement, (four people are waiting for a bus).

The small $_4$ written to the lower right of $P$ tells us how many of these objects are to be used in each arrangement, (four people getting on the bus).

Thus, ${}_4P_4 = 4! = 4 \times 3 \times 2 \times 1 = 24$.

Similarly, ${}_5P_5 = 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$.

In the next section, we will study examples where not all the objects are used in the arrangement. We will also examine a calculator key used with permutations. For now, we make the following observation:

► **For any natural number $n$, the number of permutations of $n$ objects taken $n$ at a time can be represented as:**

$$\boldsymbol{{}_nP_n = n! = n(n-1)(n-2) \times \cdots \times 3 \times 2 \times 1}$$

## EXAMPLE 1

Compute the value of each expression.

**a.** $6!$ **b.** ${}_{2}P_{2}$ **c.** $\frac{7!}{3!}$

***Solution*** **a.** $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

**b.** ${}_{2}P_{2} = 2! = 2 \times 1 = 2$

**c.** $\frac{7!}{3!} = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} = \frac{7 \times 6 \times 5 \times 4}{1} \times \frac{3 \times 2 \times 1}{3 \times 2 \times 1} = 840$

***Calculator Solution*** **a.** ENTER: 6 MATH ◀ 4 ENTER **c.** ENTER: 7 MATH ◀ 4 ENTER ÷ 3 MATH ◀ 4 ENTER

DISPLAY:

```
6!
            720
```

DISPLAY:

```
7!/3!
            840
```

*Answers* **a.** 120 **b.** 2 **c.** 840

## EXAMPLE 2

Paul wishes to call Virginia, but he has forgotten her unlisted telephone number. He knows that the exchange (the first three digits) is 555, and that the last four digits are 1, 4, 7, and 9, but he cannot remember their order. What is the maximum number of telephone calls that Paul may have to make in order to dial the correct number?

***Solution*** The telephone number is 555- . Since the last four digits will be an arrangement of 1, 4, 7, and 9, this is a permutation of four numbers, taken four at a time.

$${}_{4}P_{4} = 4! = 4 \times 3 \times 2 \times 1 = 24 \text{ possible orders}$$

*Answer* The maximum number of calls that Paul may have to make is 24.

## Permutations That Use Some of the Elements

At times, we deal with situations involving permutations in which we are given $n$ objects, but we use fewer than $n$ objects in each arrangement. For example, a teacher has announced that he will call students from the first row to explain homework problems at the board. The students in the first row are George, Helene, Jay, Karla, Lou, and Marta. If there are only two homework problems, and each problem is to be explained by a different student, in how many orders may the teacher select students to go to the board?

We know that the first problem can be assigned to any of six students. Once this problem is explained, the second problem can be assigned to any of the five remaining students. We use the counting principle to find the number of possible orders in which the selection can be made.

$$6 \times 5 = 30 \text{ possible orders}$$

If there are three problems, then after the first two students have been selected, there are four students who could be selected to explain the third problem. Extend the counting principle to find the number of possible orders in which the selection can be made.

$$6 \times 5 \times 4 = 120 \text{ possible orders}$$

Note that the starting number is the number of persons in the group from which the selection is made. Each of the factors is one less than the preceding factor. The number of factors is the number of choices to be made.

Using the language of permutations, we say that the number of permutations of six objects taken three at a time is 120.

## The Symbols for Permutations

In general, if we have a set of $n$ different objects, and we make arrangements of $r$ objects from this set, we represent the number of arrangements by the symbol ${}_nP_r$. The subscript, $r$, representing the number of factors being used, must be less than or equal to $n$, the total number of objects in the set. Thus:

▶ **For numbers $n$ and $r$, where $r \leq n$, the permutation of $n$ objects, taken $r$ at a time, is found by the formula:**

$$ {}_nP_r = \underbrace{n(n-1)(n-2)\cdots}_{r \text{ factors}} $$

This formula can also be written as:

$$ {}_nP_r = n(n-1)(n-2)\cdots(n-r+1) $$

Note that when there are $r$ factors, the last factor is $(n - r + 1)$.

In the example given above, in which three students were selected from a group of six, $n = 6$, $r = 3$, and the last factor is $n - r + 1 = 6 - 3 + 1 = 4$.

## Permutations and the Calculator

There are many ways to use a calculator to evaluate a permutation. In the three solutions presented here, we will evaluate the permutation ${}_8P_3$, the order in which, from a set of 8 elements, 3 can be selected.

**METHOD 1.** *Use the Multiplication Key,* [×].

The number permutation is simply the product of factors. In ${}_8P_3$, the first factor, 8, is multiplied by $(8 - 1)$, or 7, and then by $(8 - 2)$, or 6. Here, exactly three factors have been multiplied.

ENTER: 8 [×] 7 [×] 6 [ENTER]

DISPLAY:
```
8*7*6
                336
```

This approach can be used for any permutation. In the general permutation ${}_nP_r$, where $r \leq n$, the first factor $n$ is multiplied by $(n - 1)$, and then by $(n - 2)$, and so on, until exactly $r$ factors have been multiplied.

**METHOD 2.** *Use the Factorial Function.*

When evaluating ${}_8P_3$, it appears as if we start to evaluate 8! but then we stop after multiplying only three factors. There is a way to write the product $8 \times 7 \times 6$ using factorials. As shown below, when we divide 8! by 5!, all but the first three factors will cancel, leaving $8 \times 7 \times 6$ or 336. Since the factorial in the denominator uses all but three factors, $8 - 3 = 5$ can be used to find that factorial:

$$\begin{aligned} {}_8P_3 &= 8 \times 7 \times 6 \\ &= 8 \times 7 \times 6 \times \frac{5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1} \\ &= \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1} \\ &= \frac{8!}{5!} \\ &= \frac{8!}{(8-3)!} \end{aligned}$$

We now evaluate this fraction on a calculator.

ENTER: 8 [MATH] [◄] [4] [÷] 5 [MATH] [◄] [4] [ENTER]

DISPLAY:
```
8!/5!
                336
```

This approach shows us that there is another formula that can be used for the general permutation ${}_nP_r$, where $r < n$:

$$\boldsymbol{{}_nP_r = \frac{n!}{(n-r)!}}$$

**METHOD 3.** *Use the Permutation Function.*

Calculators have a special function to evaluate permutations. On a graphing calculator, $nPr$ is found by first pressing MATH and then using the left arrow key to highlight the PRB menu. The value of $n$ is entered first, then the $nPr$ symbol is entered followed by the value of $r$.

ENTER: 8 MATH ◄ 2 3 ENTER

DISPLAY:

```
8 nPr 3
                 336
```

## EXAMPLE 3

Evaluate ${}_6P_2$.

***Solution*** This is a permutation of six objects, taken two at a time. There are two possible formulas to use.

$$ {}_6P_2 = 6 \times 5 = 30 \qquad {}_6P_2 = \frac{6!}{4!} = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1} = 6 \times 5 = 30 $$

***Calculator Solution*** ENTER: 6 MATH ◄ 2 2 ENTER

DISPLAY:

```
6 nPr 2
                  30
```

*Answer* ${}_6P_2 = 30$

## EXAMPLE 4

How many three-letter "words" (arrangements of letters) can be formed from the letters L, O, G, I, C if each letter is used only once in a word?

***Solution*** Forming three-letter arrangements from a set of five letters is a permutation of five, taken three at a time. Thus:

$$ {}_5P_3 = 5 \times 4 \times 3 = 60 $$

or

$$ {}_5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = 5 \times 4 \times 3 = 60 $$

*Answer* 60 words

**EXAMPLE 5**

A lottery ticket contains a four-digit number. How many possible four-digit numbers are there when:

**a.** a digit may appear only once in the number?

**b.** a digit may appear more than once in the number?

***Solution*** **a.** If a digit appears only once in a four-digit number, this is a permutation of 10 digits, taken four at a time. Thus:

$$_{10}P_4 = 10 \times 9 \times 8 \times 7 = 5{,}040$$

**b.** If a digit may appear more than once, we can choose any of 10 digits for the first position, then any of 10 digits for the second position, and so forth. By the counting principle:

$$10 \times 10 \times 10 \times 10 = 10{,}000$$

*Answers* **a.** 5,040 **b.** 10,000

## EXERCISES

### Writing About Mathematics

**1.** Show that $n! = n(n - 1)!$.

**2.** Which of these two values, if either, is larger: $_9P_9$ or $_9P_8$? Explain your answer.

### Developing Skills

In 3–17, compute the value of each expression.

**3.** $4!$ **4.** $6!$ **5.** $7!$ **6.** $3! + 2!$ **7.** $(3 + 2)!$

**8.** $_3P_3$ **9.** $_8P_8$ **10.** $\frac{8!}{5!}$ **11.** $_6P_3$ **12.** $_{10}P_2$

**13.** $_{20}P_2$ **14.** $_{11}P_4$ **15.** $_7P_6$ **16.** $_{255}P_2$ **17.** $\frac{999!}{(999 - 5)!}$

In 18–21, in each case, how many three-letter arrangements can be formed if a letter is used only once?

**18.** LION **19.** TIGER **20.** MONKEY **21.** LEOPARD

**22.** Write the following expressions in the order of their values, beginning with the smallest: $_{60}P_5$, $_{45}P_6$, $_{24}P_7$, $_{19}P_7$.

## Applying Skills

**23.** Using the letters E, M, I, T: **a.** How many arrangements of four letters can be found if each letter is used only once in the "word"? **b.** List these "words."

**24.** In how many different ways can five students be arranged in a row?

**25.** In a game of cards, Gary held exactly one club, one diamond, one heart, and one spade. In how many different ways can Gary arrange these four cards in his hand?

**26.** There are nine players on a baseball team. The manager must establish a batting order for the players at each game. The pitcher will bat last. How many different batting orders are possible for the eight remaining players on the team?

**27.** There are 30 students in a class. Every day the teacher calls on different students to write homework problems on the board, with each problem done by only one student. In how many ways can the teacher call students to the board if the homework consists of:

**a.** only 1 problem? **b.** 2 problems? **c.** 3 problems?

**28.** At the Olympics, three medals are given for each competition: gold, silver, and bronze. Tell how many possible winning orders there are for the gymnastic competition if the number of competitors is:

**a.** 7 **b.** 9 **c.** 11 **d.** $n$

**29.** How many different ways are there to label the three vertices of a scalene triangle, using no letter more than once, when:

**a.** we use the letters $R$, $S$, $T$?

**b.** we use all the letters of the English alphabet?

**30.** A class of 31 students elects four people to office, namely, a president, vice president, secretary, and treasurer. In how many possible ways can four people be elected from this class?

**31.** How many possible ways are there to write two initials, using the letters of the English alphabet, if:

**a.** an initial may appear only once in each pair?

**b.** the same initial may be used twice?

In 32–35: **a.** Write each answer in factorial form. **b.** Write each answer, after using a calculator, in scientific notation.

**32.** In how many different orders can 60 people line up to buy tickets at a theater?

**33.** We learn the alphabet in a certain order, starting with A, B, C, and ending with Z. How many possible orders are there for listing the letters of the English alphabet?

**34.** Twenty-five people are waiting for a bus. When the bus arrives, there is room for 18 people to board. In how many ways could 18 of the people who are waiting board the bus?

**35.** Forty people attend a party at which eight door prizes are to be awarded. In how many orders can the names of the winners be announced?

## 15-10 PERMUTATIONS WITH REPETITION

How many different "words" or arrangements of four letters can be formed using each letter of the word PEAK?

This is the number of permutations of four things, taken four at a time. Since ${}_4P_4 = 4! = 4 \times 3 \times 2 \times 1 = 24$, there are 24 possible words or arrangements.

| | | | | | |
|---|---|---|---|---|---|
| PEAK | PAEK | EAPK | AEPK | EPAK | APEK |
| PAKE | PEKA | APKE | EPKA | AEKP | EAKP |
| PKAE | PKEA | AKPE | EKPA | AKEP | EKAP |
| KPAE | KPEA | KAEP | KEAP | KAPE | KEPA |

Now consider a related question. How many different words or arrangements of four letters can be formed using each letter of the word PEEK?

This is an example of a permutation with repetition because the letter E is repeated in the word. We can try to list the different arrangements by simply replacing the A in each of the arrangements given for the word PEAK. Let the E from PEAK be $E_1$ and the E that replaces A be $E_2$. The arrangements can be written as follows:

| | | | | | |
|---|---|---|---|---|---|
| $PE_1E_2K$ | $PE_2E_1K$ | $E_1E_2PK$ | $E_2E_1PK$ | $E_1PE_2K$ | $E_2PE_1K$ |
| $PE_2KE_1$ | $PE_1KE_2$ | $E_2PKE_1$ | $E_1PKE_2$ | $E_2E_1KP$ | $E_1E_2KP$ |
| $PKE_2E_1$ | $PKE_1E_2$ | $E_2KPE_1$ | $E_1KPE_2$ | $E_2KE_1P$ | $E_1KE_2P$ |
| $KPE_2E_1$ | $KPE_1E_2$ | $KE_2E_1P$ | $KE_1E_2P$ | $KE_2PE_1$ | $KE_1PE_2$ |

We have 24 different arrangements if we consider $E_1$ to be different from $E_2$. But they are not really different. Notice that if we consider the E's to be the same, every word in the first column is the same as a word in the second column, every word in the third column is the same as a word in the fourth column, and every word in the fifth column is the same as a word in the sixth column. Therefore, only the first, third, and fifth columns are different and there are $\frac{24}{2}$ or 12 arrangements of four letters when two of them are the same. This is the number of arrangements of four letters divided by the number of arrangements of two letters.

Now consider a third word. In how many ways can the letters of EEEK be arranged? We will use the 24 arrangements of the letters of $PE_1E_2K$ and write $E_3$ in place of P.

| | | | | | |
|---|---|---|---|---|---|
| $E_3E_1E_2K$ | $E_3E_2E_1K$ | $E_1E_2E_3K$ | $E_2E_1E_3K$ | $E_1E_3E_2K$ | $E_2E_3E_1K$ |
| $E_3E_2KE_1$ | $E_3E_1KE_2$ | $E_2E_3KE_1$ | $E_1E_3KE_2$ | $E_2E_1KE_3$ | $E_1E_2KE_3$ |
| $E_3KE_2E_1$ | $E_3KE_1E_2$ | $E_2KE_3E_1$ | $E_1K\,E_3E_2$ | $E_2KE_1E_3$ | $E_1KE_2E_3$ |
| $KE_3E_2E_1$ | $K\,E_3E_1E_2$ | $KE_2E_1E_3$ | $KE_1E_2E_3$ | $KE_2E_3E_1$ | $KE_1E_3E_2$ |

Notice that each row is the same arrangement if we consider all the E's to be the same letter. The 4! arrangements of four letters are in groups of 3! or 6, the number of different orders in which the 3 E's can be arranged among themselves.

Therefore the number of possible arrangements is:

$$\frac{4!}{3!} = \frac{4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} = \frac{24}{6} = 4$$

If we consider all E's to be the same, the four arrangements are:

EEEK EEKE EKEE KEEE

In general, the number of permutations of $n$ things, taken $n$ at a time, with $r$ of these things identical, is:

$$\frac{n!}{r!}$$

## EXAMPLE 1

How many six-digit numerals can be written using all of the following digits: 2, 2, 2, 2, 3, and 5?

***Solution*** This is the number of permutations of six things taken six at a time, with 4 of the digits 2, 2, 2, 2 identical. Therefore:

$$\frac{6!}{4!} = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1} = 6 \times 5 = 30$$

***Calculator Solution*** ENTER: 6 MATH ◀ 4 ÷ 4 MATH ◀ 4 ENTER

DISPLAY:

```
6!/4!
                30
```

*Answer* 30 six-digit numerals

## EXAMPLE 2

Three children, Rita, Ann, and Marie, take turns doing the dishes each night of the week. At the beginning of each week they make a schedule. If Rita does the dishes three times, and Ann and Marie each do them twice, how many different schedules are possible?

***Solution*** We can think of this as an arrangement of the letters RRRAAMM, that is, an arrangement of seven letters (for the seven days of the week) with R appearing three times and A and M each appearing twice. Therefore we will divide 7! by 3!, the number of arrangements of Rita's days; then by 2!, the number of arrangements of Ann's days; and finally by 2! again, the number of arrangements of Marie's days.

$$\text{Number of arrangements} = \frac{7!}{3! \times 2! \times 2!} = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 2 \times 1 \times 2 \times 1} = 210$$

*Answer* 210 possible schedules

## EXERCISES

### Writing About Mathematics

**1. a.** List the six different arrangements or permutations of the letters in the word TAR.

**b.** Explain why exactly six arrangements are possible.

**2. a.** List the three different arrangements or permutations of the letters in the word TOT.

**b.** Explain why exactly three arrangements are possible.

### Developing Skills

In 3–6, how many five-letter permutations are there of the letters of each given word?

**3.** APPLE **4.** ADDED **5.** VIVID **6.** TESTS

In 7–14: **a.** How many different six-letter arrangements can be written using the letters in each given word? **b.** How many different arrangements begin with E? **c**. If an arrangement is chosen at random, what is the probability that it begins with E?

**7.** SIMPLE **8.** FREEZE **9.** SYSTEM **10.** BETTER

**11.** SEEDED **12.** DEEDED **13.** TATTOO **14.** ELEVEN

In 15–22, find the number of distinct arrangements of the letters in each word.

**15.** STREETS **16.** INSISTS **17.** ESTEEMED **18.** DESERVED

**19.** TENNESSEE **20.** BOOKKEEPER **21.** MISSISSIPPI **22.** UNUSUALLY

In 23–26, in each case find: **a.** How many different five-digit numerals can be written using all five digits listed? **b.** How many of the numerals formed from the given digits are greater than 12,000 and less than 13,000? **c.** If a numeral formed from the given digits is chosen at random, what is the probability that it is greater than 12,000 and less than 13,000?

**23.** 1, 2, 3, 4, 5 **24.** 1, 2, 2, 2, 2 **25.** 1, 1, 2, 2, 2 **26.** 2, 2, 2, 2, 3

In 27–31, when written without using exponents, $a^2x$ can be written as *aax*, *axa*, or *xaa*. How many different arrangements of letters are possible for each given expression when written without exponents?

**27.** $b^3y$ **28.** $a^2b^5$ **29.** $abx^6$ **30.** $a^2by^7$ **31.** $a^4b^8$

### Applying Skills

**32.** A bookseller has 7 copies of a novel and 3 copies of a biography. In how many ways can these 10 books be arranged on a shelf?

**33.** In how many ways can 6 white flags and 3 blue flags be arranged one above another on a single rope on a flagstaff?

**34.** Florence has 6 blue beads, 8 white beads, and 4 green beads, all the same size. In how many ways can she string these beads on a chain to make a necklace?

**35.** Frances has 8 tulip bulbs, 10 daffodil bulbs, and 7 crocus bulbs. In how many ways can Frances plant these bulbs in a border along the edge of her garden?

**36.** Anna has 2 dozen Rollo bars and 1 dozen apples as treats for Halloween. In how many ways can Anna hand out 1 treat to each of 36 children who come to her door?

**37.** A dish of mixed nuts contains 7 almonds, 5 cashews, 3 filberts, and 4 peanuts. In how many different orders can Jerry eat the contents of the dish, one nut at a time?

**38.** Print your first and last names using capital letters. How many different arrangements of the letters in your full name are possible?

## 15-11 COMBINATIONS

### Comparing Permutations and Combinations

A **combination** is a collection of things in which order is not important. Before we discuss combinations, let us start with a problem we know how to solve.

Ann, Beth, Carlos, and Dava are the only members of a school club. In how many ways can they elect a president and a treasurer for the club?

Any one of the 4 students can be elected as president. After this happens, any one of the 3 remaining students can be elected treasurer. Thus there are $4 \times 3$ or 12 possible outcomes. Using the initials to represent the students involved, we can write these 12 arrangements:

(A, B) (B, A) (B, C) (C, B)
(A, C) (C, A) (B, D) (D, B)
(A, D) (D, A) (C, D) (D, C)

These are the permutations. We could have found that there are 12 permutations by using the formula:

$$_4P_2 = 4 \times 3 = 12$$

*Answer:* 12 permutations

Now let us consider two problems of a different type involving the members of the same club.

Ann, Beth, Carlos, and Dava are the only members of a school club. In how many ways can they choose two members to represent the club at a student council meeting?

If we look carefully at the list of 12 possible selections given in the answer to problem 1, we can see that while (A, B) and (B, A) are two different choices for president and treasurer, sending Ann and Beth to the student council meeting is exactly the same as sending Beth and Ann. For this problem let us match up answers that consist of the same two persons.

$$(A, B) \leftrightarrow (B, A) \qquad (B, C) \leftrightarrow (C, B)$$
$$(A, C) \leftrightarrow (C, A) \qquad (B, D) \leftrightarrow (D, B)$$
$$(A, D) \leftrightarrow (D, A) \qquad (C, D) \leftrightarrow (D, C)$$

Although order is important in listing slates of officers in problem 1, there is no reason to consider the order of elements in this problem. In fact, if we think of two representatives to the student council as a *set* of two club members, we can list the sets of representatives as follows:

$$\{A, B\} \qquad \{B, C\}$$
$$\{A, C\} \qquad \{B, D\}$$
$$\{A, D\} \qquad \{C, D\}$$

From this list we can find the number of *combinations* of 4 things, taken 2 at a time, written in symbols as ${}_4C_2$. The answer to this problem is found by dividing the number of permutations of 4 things taken 2 at a time by 2!. Thus:

$${}_4C_2 = \frac{{}_4P_2}{2!} = \frac{4 \times 3}{2 \times 1} = \frac{12}{2} = 6$$

*Answer:* 6 combinations

Ann, Beth, Carlos, and Dava are the only members of a school club. In how many ways can they choose a three-member committee to work on the club's next project? Is order important to this answer? If 3 officers were to be elected, such as a president, a treasurer, and a secretary, then order would be important and the number of permutations would be needed. However, a committee is a set of people. In listing the elements of a set, order is not important. Compare the permutations and combinations of 4 persons taken 3 at a time:

| *Permutations* | | | | | | *Combinations* |
|---|---|---|---|---|---|---|
| (A, B, C) | (A, C, B) | (B, A, C) | (B, C, A) | (C, A, B) | (C, B, A) | {A, B, C} |
| (A, B, D) | (A, D, B) | (B, A, D) | (B, D, A) | (D, A, B) | (D, B, A) | {A, B, D} |
| (A, C, D) | (A, D, C) | (C, A, D) | (C, D, A) | (D, A, C) | (D, C, A) | {A, C, D} |
| (B, C, D) | (B, D, C) | (C, B, D) | (C, D, B) | (D, B, C) | (D, C, B) | {B, C, D} |

While there are 24 permutations, written as ordered triples, there are only 4 combinations, written as sets. For example, in the first row of permutations, there are 3! or $3 \times 2 \times 1$ or 6 ordered triples (slates of officers) including Ann, Beth, and Carlos.

However, there is only one set (committee) that includes these three persons. Therefore, the number of ways to select a three-person committee from a group of four persons is the number of *combinations* of 4 things taken 3 at a time. This number is found by dividing the number of permutations of 4 things taken 3 at a time by 3!, the number of arrangements of the three things.

$${}_4C_3 = \frac{{}_4P_3}{3!} = \frac{4 \times 3 \times 2}{3 \times 2 \times 1} = \frac{24}{6} = 4$$

*Answer:* 4 combinations

In general, for counting numbers $n$ and $r$, where $r \leq n$, the number of combinations of $n$ things taken $r$ at a time is found by using the formula:

$${}_nC_r = \frac{{}_nP_r}{r!}$$

On a graphing calculator, the sequence of keys needed to find ${}_nC_r$ is similar to that for ${}_nP_r$. The combination symbol is entry 3 in the PRB menu.

ENTER: 4 **MATH** **◄** **3** 3 **ENTER**

DISPLAY:

```
4 nCr 3
                4
```

**Note:** The notation $\binom{n}{r}$ also represents the number of combinations of $n$ things taken $r$ at a time. Thus:

$$\binom{n}{r} = {}_nC_r \quad \text{or} \quad \binom{n}{r} = \frac{{}_nP_r}{r!}$$

## Some Relationships Involving Combinations

Given a group of 5 people, how many different 5-person committees can be formed? Common sense tells us that there is only 1 such committee, namely, the committee consisting of all 5 people. Using combinations, we see that

$${}_5C_5 = \frac{{}_5P_5}{5!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1} = 1$$

Also,

$${}_3C_3 = \frac{{}_3P_3}{3!} = \frac{3 \times 2 \times 1}{3 \times 2 \times 1} = 1 \qquad \text{and} \qquad {}_4C_4 = \frac{{}_4P_4}{4!} = \frac{4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1} = 1$$

► **For any counting number $n$, ${}_nC_n = 1$.**

Given a group of 5 people, in how many different ways can we select a committee consisting of no people, or 0 people? Common sense tells us that there is only 1 way to select no one. Thus, using combinations, ${}_5C_0 = 1$. Let us agree to the following generalization:

► **For any counting number $n$, ${}_nC_0 = 1$.**

In how many ways can we select a committee of 2 people from a group of 7 people? Since a committee is a combination,

$$_7C_2 = \frac{_7P_2}{2!} = \frac{7 \times 6}{2 \times 1} = 21.$$

Now, given a group of 7 people, in how many ways can 5 people *not* be appointed to the committee? Since each set of people not appointed is a combination:

$$_7C_5 = \frac{_7P_5}{5!} = \frac{7 \times 6 \times 5 \times 4 \times 3}{5 \times 4 \times 3 \times 2 \times 1} = \frac{7 \times 6}{2 \times 1} \times \frac{5 \times 4 \times 3}{5 \times 4 \times 3} = \frac{7 \times 6}{2 \times 1} \times 1 = 21$$

Notice that $_7C_2 = {_7C_5}$. In other words, starting with a group of 7 people, the number of sets of 2 people that can be selected is equal to the number of sets of 5 people that can be *not* selected. In the same way it can be shown that $_7C_3 = {_7C_4}$, that $_7C_6 = {_7C_1}$, and that $_7C_7 = {_7C_0}$.

In general, starting with $n$ objects, the number of ways to choose $r$ objects for a combination is equal to the number of ways to *not* choose $(n - r)$ objects for the combination.

► **For whole numbers $n$ and $r$, where $r \leq n$,**

$$_nC_r = {_nC_{n-r}}$$

**KEEP IN MIND**

| PERMUTATIONS | COMBINATIONS |
|---|---|
| **1.** *Order is important.*<br>Think of ordered elements such as ordered pairs and ordered triples.<br>**2.** An *arrangement* or *a slate of officers* indicates a permutation. | **1.** *Order is not important.*<br>Think of sets.<br>**2.** A *committee*, or a *selection of a group*, indicates a combination. |

## EXAMPLE 1

Evaluate: $_{10}C_3$

***Solution*** This is the number of combinations of 10 things, taken 3 at a time. Thus

$$_{10}C_3 = \frac{_{10}P_3}{3!} = \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = \frac{720}{6} = 120$$

***Calculator Solution*** ENTER: 10 MATH ◄ 3 3 ENTER

DISPLAY:

```
10 nCr 3
                120
```

*Answer* 120

## EXAMPLE 2

Evaluate: $\binom{25}{23}$

***Solution*** (1) This is an alternative form for the number of combinations of 25 things, taken 23 at a time: $\binom{25}{23} = {}_{25}C_{23}$.

(2) Since ${}_nC_r = {}_nC_{n-r}$, ${}_{25}C_{23} = {}_{25}C_2$.

(3) Using ${}_{25}C_2$, perform the shorter computation. Thus:

$$\binom{25}{23} = {}_{25}C_{23} = {}_{25}C_2 = \frac{25 \times 24}{2 \times 1} = 300$$

*Answer* 300

## EXAMPLE 3

There are 10 teachers in the science department. How many 4-person committees can be formed in the department if Mrs. Martens and Dr. Blumenthal, 2 of the teachers, must be on each committee?

***Solution*** Since Mrs. Martens and Dr. Blumenthal must be on each committee, the problem becomes one of filling 2 positions on a committee from the remaining 8 teachers.

$${}_8C_2 = \frac{{}_8P_2}{2!} = \frac{8 \times 7}{2 \times 1} = 28$$

*Answer* 28 committees

## EXAMPLE 4

There are six points in a plane, no three of which are collinear. How many straight lines can be drawn using pairs of these three points?

***Solution*** Whether joining points $A$ and $B$, or points $B$ and $A$, only 1 line exists, namely, $\overleftrightarrow{AB}$. Since order is not important here, this is a combination of 6 points, taken 2 at a time.

$${}_6C_2 = \frac{{}_6P_2}{2!} = \frac{6 \times 5}{2 \times 1} = 15$$

*Answer* 15 lines

## EXAMPLE 5

Lisa Dwyer is a teacher at a local high school. In her class, there are 10 boys and 20 girls. Find the number of ways in which Ms. Dwyer can select a team of 3 students from the class to work on a group project if the team consists of:

**a.** any 3 students **b.** 1 boy and 2 girls **c.** 3 girls **d.** at least 2 girls

***Solution*** **a.** The class contains 10 boys and 20 girls, for a total of 30 students. Since order is not important on a team, this is a combination of 30 students, taken 3 at a time.

$$_{30}C_3 = \frac{_{30}P_3}{3!} = \frac{30 \times 29 \times 28}{3 \times 2 \times 1} = 4{,}060$$

**b.** This is a compound event. To find the number of ways to select 1 boy out of 10 boys for a team, use $_{10}C_1$. To find the number of ways to select 2 girls out of 20 for the team, use $_{20}C_2$. Then, by the counting principle, multiply the results.

$$_{10}C_1 \times {}_{20}C_2 = \frac{10}{1} \times \frac{20 \times 19}{2 \times 1} = 10 \times 190 = 1{,}900$$

**c.** This is another compound event, in which 0 boys out of 10 boys and 3 girls out of 20 girls are selected. Recall that $_{10}C_0 = 1$. Thus:

$$_{10}C_0 \times {}_{20}C_3 = 1 \times \frac{20 \times 19 \times 18}{3 \times 2 \times 1} = 1{,}140$$

Note that this could also have been thought of as the simple event of selecting 3 girls out of 20 girls:

$$_{20}C_3 = \frac{20 \times 19 \times 18}{3 \times 2 \times 1} = 1{,}140$$

**d.** A team of at least 2 girls can consist of exactly 2 girls (see part **b**) or exactly 3 girls (see part **c**). Since these events are disjoint, add the solutions to parts **b** and **c**:

$$1{,}900 + 1{,}140 = 3{,}040$$

*Answers* **a.** 4,060 teams **b.** 1,900 teams **c.** 1,140 teams **d.** 3,040 teams

## EXERCISES

### Writing About Mathematics

**1.** Explain the difference between a permutation and a combination.

**2.** A set of $r$ letters is to be selected from the 26 letters of the English alphabet. For what value of $r$ is the number of possible sets of numbers greatest?

## Developing Skills

In 3–14, evaluate each expression.

**3.** ${}_{15}C_2$ **4.** ${}_{12}C_3$ **5.** ${}_{10}C_4$ **6.** ${}_{25}C_1$

**7.** ${}_{13}C_0$ **8.** ${}_{14}C_{14}$ **9.** ${}_{9}C_8$ **10.** ${}_{200}C_{198}$

**11.** $\binom{7}{3}$ **12.** $\binom{9}{4}$ **13.** $\binom{17}{17}$ **14.** $\binom{499}{2}$

**15.** Find the number of combinations of 6 things, taken 3 at a time.

**16.** How many different committees of 3 people can be chosen from a group of 9 people?

**17.** How many different subsets of exactly 7 elements can be formed from the set

$$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}?$$

**18.** For each given number of non-collinear points in a plane, how many straight lines can be drawn?

**a.** 3 **b.** 4 **c.** 5 **d.** 7 **e.** 8 **f.** $n$

**19.** Consider the following formulas:

(1) ${}_{n}C_r = \frac{{}_{n}P_r}{r!}$ (2) ${}_{n}C_r = \frac{n!}{(n-r)!r!}$ (3) ${}_{n}C_r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$

**a.** Evaluate ${}_{8}C_3$ using each of the three formulas.

**b.** Evaluate ${}_{11}C_7$ using each of the three formulas.

**c.** Can all three formulas be used to find the combination of $n$ things, taken $r$ at a time?

## Applying Skills

**20.** A coach selects players for a team. If, while making this first selection, the coach pays no attention to the positions that individuals will play, how many teams are possible?

**a.** Of 14 candidates, Coach Richko needs 5 for a basketball team.

**b.** Of 16 candidates, Coach Jones needs 11 for a football team.

**c.** Of 13 candidates, Coach Greves needs 9 for a baseball team.

**21.** A disc jockey has 25 recordings at hand, but has time to play only 22 on the air. How many sets of 22 recordings can be selected?

**22.** There are 14 teachers in a mathematics department.

**a.** How many 4-person committees can be formed in the department?

**b.** How many 4-person committees can be formed if Mr. McDonough, 1 of the 14, must be on the committee?

**c.** How many 4-person committees can be formed if Mr. Goldstein and Mrs. Friedel, 2 of the 14, must be on the committee?

**23.** There are 12 Republicans and 10 Democrats on a senate committee. From this group, a 3-person subcommittee is to be formed. Find the number of 3-person subcommittees that consist of:

**a.** any members of the senate committee
**b.** Democrats only
**c.** 1 Republican and 2 Democrats
**d.** at least 2 Democrats
**e.** John Clark, who is a Democrat, and any 2 Republicans

**24.** Sue Bartling loves to read mystery books and car-repair manuals. On a visit to the library, Sue finds 9 new mystery books and 3 car-repair manuals. She borrows 4 of these books. Find the number of different sets of 4 books Sue can borrow if:

**a.** all are mystery books
**b.** exactly 2 are mystery books
**c.** only 1 is a mystery book
**d.** all are car-repair manuals

**25.** Cards are drawn at random from a 52-card deck. Find the number of different 5-card poker hands possible consisting of:

**a.** any 5 cards from the deck
**b.** 3 aces and 2 kings
**c.** 4 queens and any other card
**d.** 5 spades
**e.** 2 aces and 3 picture cards
**f.** 5 jacks

**26.** How many committees consisting of 7 people or more can be formed from a group of 10 people?

**27.** There are 12 roses growing in Heather's garden. How many different ways can Heather choose roses for a bouquet consisting of more than 8 roses?

## 15-12 PERMUTATIONS, COMBINATIONS, AND PROBABILITY

In this section, a variety of probability questions are presented. In some cases, permutations should be used. In other cases, combinations should be used. When answering these questions, the following should be kept in mind:

1. If the question asks "How many?" or "In how many ways?" the answer will be a whole number.
2. If the question asks "What is the probability?" the answer will be a value between 0 and 1 inclusive.
3. If order is important, use permutations.
4. If order is not important, use combinations.

In Examples 1–4, Ms. Fenstermacher must select 4 students to represent the class in a spelling bee. Her best students include 3 girls (Callie, Daretta, and Jessica) and 4 boys (Bandu, Carlos, Sanjit, and Uri).

## EXAMPLE 1

Ms. Fenstermacher decides to select the 4 students from the 7 best by drawing names from a hat. How many different groups of 4 are possible?

**Solution** In choosing a group of 4 students out of 7, order is not important. Therefore, use combinations.

$$_7C_4 = \frac{_7P_4}{4!} = \frac{7 \times 6 \times 5 \times 4}{4 \times 3 \times 2 \times 1} = 35$$

*Answer* 35 groups

## EXAMPLE 2

What is the probability that the 4 students selected for the spelling bee will consist of 2 girls and 2 boys?

**Solution** (1) The answer to Example 1 shows 35 possible groups, or $n(S) = 35$.

(2) Since order is not important in choosing groups, use combinations.

The number of ways to choose 2 girls out of 3 is $_3C_2 = \frac{3 \times 2}{2 \times 1} = 3$.

The number of ways to choose 2 boys out of 4 is $_4C_2 = \frac{4 \times 3}{2 \times 1} = 6$.

Event $E$ is the compound event of choosing 2 girls and 2 boys, or

$$n(E) = {_3C_2} \times {_4C_2} = 3 \times 6 = 18.$$

(3) Thus $P(E) = \frac{_3C_2 \times {_4C_2}}{_7C_4} = \frac{18}{35}$.

*Answer* $P(\text{2 girls, 2 boys}) = \frac{18}{35}$

## EXAMPLE 3

The students chosen are Callie, Jessica, Carlos, and Sanjit. In how many orders can these 4 students be called upon in the spelling bee?

**Solution** The number of ways in which 4 students can be called upon in a spelling bee means that someone is first, someone else is second, and so on. Since this is a problem about order, use permutations.

$$_4P_4 = 4! = 4 \times 3 \times 2 \times 1 = 24$$

*Answer* 24 orders

## EXAMPLE 4

For the group consisting of Callie, Jessica, Carlos, and Sanjit, what is the probability that the first 2 students called upon will be girls?

***Solution*** This question may be answered using various methods, two of which are shown below.

**METHOD 1.** *Counting Principle*

There are 2 girls out of 4 students who may be called first. Once a girl is called upon, only 1 girl remains in the 3 students not yet called upon. Apply the counting principle of probability.

$$P(\text{first 2 are girls}) = P(\text{girl first}) \times P(\text{girl second}) = \frac{2}{4} \times \frac{1}{3} = \frac{2}{12} = \frac{1}{6}$$

**METHOD 2.** *Permutations*

Let $n(S)$ equal the number of ways to call 2 of the 4 students in order, and let $n(E)$ equal the number of ways to call 2 of the 2 girls in order.

$$P(E) = \frac{n(E)}{n(S)} = \frac{{}_2P_2}{{}_4P_2} = \frac{2 \times 1}{4 \times 3} = \frac{2}{12} = \frac{1}{6}$$

*Answer* $P(\text{first 2 are girls}) = \frac{1}{6}$

## EXERCISES

### Writing About Mathematics

**1.** A committee of three persons is to be chosen from a group of eight persons. Olivia is one of the persons in that group. Olivia said that since ${}_8C_3 = {}_8C_5$, the probability that she will be chosen for that committee is equal to the probability that she will not be chosen. Do you agree with Olivia? Explain why or why not.

**2.** Four letters are to be selected at random from the alphabet. Jenna found the probability that the four letters followed S in alphabetical order by using permutations. Colin found the probability that the four letters followed S in alphabetical order by using combinations. Who was correct? Explain your answer.

### Applying Skills

**3.** The Art Club consists of 4 girls (Jennifer, Anna, Gloria, Teresa) and 2 boys (Mark and Dan).

**a.** In how many ways can the club elect a president and a treasurer?

**b.** Find the probability that the 2 officers elected are both girls.

**c.** How many 2-person teams can be selected to work on a project?

**d.** Find the probability that a 2-person team consists of:
(1) 2 girls (2) 2 boys (3) 1 girl and 1 boy (4) Anna and Mark

**4.** A committee of 4 is to be chosen at random from 4 men and 3 women.

**a.** How many different 4-member committees are possible?

**b.** How many 4-member committees contain 3 men and 1 woman?

**c.** What is the probability that a 4-member committee will contain exactly 1 woman?

**d.** What is the probability that a man will be on the committee?

**e.** What is the probability that the fourth person chosen is a woman given that 3 men have already been chosen?

**5.** A committee of 6 people is to be chosen from 9 available people.

**a.** How many 6-person committees can be chosen?

**b.** The committee, when chosen, has 4 students and 2 teachers. Find the probability that a 3-person subcommittee from this group includes:

(1) students only

(2) exactly 1 teacher

(3) at least 2 students

(4) a teacher given that 2 students have been chosen

**6.** A box of chocolate-covered candies contains 7 caramels and 3 creams all exactly the same in appearance. Jim selects 4 pieces of candy.

**a.** Find the number of selections possible of 4 pieces of candy that include:

(1) 4 caramels

(2) 1 caramel and 3 creams

(3) 2 caramels and 2 creams

(4) any 4 pieces

**b.** Find the probability that Jim's selection included:

(1) 4 caramels

(2) 1 caramel and 3 creams

(3) 2 caramels and 2 creams

(4) no caramels

(5) a second cream given that 2 caramels and a cream have been selected

**7.** Two cards are drawn at random from a 52-card deck without replacement. Find the probability of drawing:

**a.** the ace of clubs and jack of clubs in either order

**b.** a red ace and a black jack in either order

**c.** 2 jacks

**d.** 2 clubs

**e.** an ace and a jack in either order

**f.** an ace and a jack in that order

**g.** a heart given that the king of hearts was drawn

**h.** a king given that a queen was drawn

**8.** Mrs. Carberry has 4 quarters and 3 nickels in her purse. If she takes 3 coins out of her purse without looking at them, find the probability that the 3 coins are worth:

**a.** exactly 75 cents

**b.** exactly 15 cents

**c.** exactly 35 cents

**d.** exactly 55 cents

**e.** more than 10 cents

**f.** less than 40 cents

**9.** A 3-digit numeral is formed by selecting digits at random from {2, 4, 6, 7} without repetition. Find the probability that the number formed:

**a.** is less than 700
**b.** is greater than 600
**c.** contains only even digits
**d.** is an even number

**10.** **a.** There are 10 runners on the track team. If 4 runners are needed for a relay race, how many different relay teams are possible?

**b.** Once the relay team is chosen, in how many different orders can the 4 runners run the race?

**c.** If Nicolette is on the relay team, what is the probability that she will lead off the race?

**11.** Lou Grant is an editor at a newspaper employing 10 reporters and 3 photographers.

**a.** If Lou selects 2 reporters and 1 photographer to cover a story, from how many possible 3-person teams can he choose?

**b.** If Lou hands out 1 assignment per reporter, in how many ways can he assign the first 3 stories to his 10 reporters?

**c.** If Lou plans to give the first story to Rossi, a reporter, in how many ways can he now assign the first 3 stories?

**d.** If 3 out of 10 reporters are chosen at random to cover a story, what is the probability that Rossi is on this team?

**12.** Chris, Willie, Tim, Matt, Juan, Bob, and Steven audition for roles in the school play.

**a.** If 2 male roles in the play are those of the hero and the clown, in how many ways can the director select 2 of the 7 boys for these roles?

**b.** Chris and Willie got the 2 leading male roles.

(1) In how many ways can the director select a group of 3 of the remaining 5 boys to work in a crowd scene?

(2) How many of these groups of 3 will include Tim?

(3) Find the probability that Tim is in the crowd scene.

**13.** There are 8 candidates for 3 seats in the student government. The candidates include 3 boys (Alberto, Peter, Thomas) and 5 girls (Elizabeth, Maria, Joanna, Rosa, Danielle). If all candidates have an equal chance of winning, find the probability that the winners include:

**a.** 3 boys
**b.** 3 girls
**c.** 1 boy and 2 girls
**d.** at least 2 girls
**e.** Maria, Peter, and anyone else
**f.** Danielle and any other two candidates
**g.** Alberto, Elizabeth, and Rosa
**h.** Rosa, Peter, and Thomas

**14.** A gumball machine contains 6 lemon-, 4 lime-, 3 cherry-, and 2 orange-flavored gumballs. Five coins are put into the machine, and 5 gumballs are obtained.

**a.** How many different sets of 5 gumballs are possible?

**b.** How many of these will contain 2 lemon and 3 lime gumballs?

**c.** Find the probability that the 5 gumballs dispensed by the machine include:

(1) 2 lemon and 3 lime
(2) 3 cherry and 2 orange
(3) 2 lemon, 2 lime, and 1 orange
(4) lemon only
(5) lime only
(6) no lemon

**15.** The letters in the word HOLIDAY are rearranged at random.

**a.** How many 7-letter words can be formed?

**b.** Find the probability that the first 2 letters are vowels.

**c.** Find the probability of no vowels in the first 3 letters.

**16.** At a bus stop, 5 people enter a bus that has only 3 empty seats.

**a.** In how many different ways can 3 of the 5 people occupy these empty seats?

**b.** If Mrs. Costa is 1 of the 5 people, what is the probability that she will *not* get a seat?

**c.** If Ann and Bill are 2 of the 5 people, what is the probability that they both will get seats?

## CHAPTER SUMMARY

**Probability** is a branch of mathematics in which the chance of an event happening is assigned a numerical value that predicts how likely that event is to occur.

**Empirical probability** may be defined as the most accurate scientific estimate, based on a large number of trials, of the cumulative relative frequency of an event happening.

An **outcome** is a result of some activity or experiment. A **sample space** is a set of all possible outcomes for the activity. An **event** is a subset of the sample space.

The **theoretical probability** of an event is the number of ways that the event can occur, divided by the total number of possibilities in the sample space. If $P(E)$ represents the probability of event $E$, $n(E)$ represents the number of outcomes in event $E$ and $n(S)$ represents the number of outcomes in the sample space $S$. The formula, which applies to **fair and unbiased** objects and situations is:

$$P(E) = \frac{n(E)}{n(S)}$$

The probability of an **impossible** event is 0.

The probability of an event that is **certain** to occur is 1.

The probability of any event $E$ must be equal to or greater than 0, and less than or equal to 1:

$$0 \leq P(E) \leq 1$$

For the shared outcomes of events $A$ and $B$:

$$P(A \text{ and } B) = \frac{n(A \cap B)}{n(S)}$$

If two events $A$ and $C$ are **mutually exclusive**:

$$P(A \text{ or } C) = P(A) + P(C)$$

If two events $A$ and $B$ are not mutually exclusive:

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

For any event $A$:

$$P(A) + P(\text{not } A) = 1$$

The sum of the probabilities of all possible singleton outcomes for any sample space must always equal 1.

**The Counting Principle:** If one activity can occur in any of $m$ ways and, following this, a second activity can occur in any of $n$ ways, then both activities can occur in the order given in $m \times n$ ways.

**The Counting Principle for Probability:** $E$ and $F$ are independent events. The probability of event $E$ is $m$ $(0 \leq m \leq 1)$ and the probability of event $F$ is $n$ $(0 \leq n \leq 1)$, the probability of the event in which $E$ and $F$ occur jointly is the product $m \times n$.

When the result of one activity in no way influences the result of a second activity, the results of these activities are called **independent events**. Two events are called **dependent events** when the result of one activity influences the result of a second activity.

If $A$ and $B$ are two events, **conditional probability** is the probability of $B$ given that $A$ has occurred. The notation for conditional probability is $P(B \mid A)$.

$$P(B \mid A) = \frac{P(A \text{ and } B)}{P(A)}$$

The general result $P(A \text{ and } B) = P(A) \times P(B \mid A)$ is true for both dependent and independent events since for independent events, $P(B \mid A) = P(B)$.

A **permutation** is an arrangement of objects in some specific order. The number of permutations of $n$ objects taken $n$ at a time, ${}_nP_n$, is equal to ***n* factorial:**

$${}_nP_n = n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$$

The number of permutations of $n$ objects taken $r$ at a time, where $r \leq n$, is equal to:

$${}_nP_r = \underbrace{n(n-1)(n-2)\cdots}_{r \text{ factors}} \qquad \text{or} \qquad {}_nP_r = \frac{n!}{(n-r)!}$$

A permutation of $n$ objects taken $n$ at a time in which $r$ are identical is equal to $\frac{n!}{r!}$.

A **combination** is a set of objects in which order is not important, as in a committee. The formula for the number of combinations of $n$ objects taken $r$ at a time $(r \leq n)$ is

$$_nC_r = \frac{_nP_r}{r!}$$

From this formula, it can be shown that $_nC_n = 1$, $_nC_0 = 1$, and $_nC_r = {}_nC_{n-r}$.

Permutations and combinations are used to evaluate $n(S)$ and $n(E)$ in probability problems.

## VOCABULARY

**15-1** Probability • Empirical study • Relative frequency • Cumulative relative frequency • Converge • Empirical probability • Trial • Experiment • Fair and unbiased objects • Biased objects • Die • Standard deck of cards

**15-2** Outcome • Sample space • Event • Favorable event • Unfavorable event • Singleton event • Theoretical probability • Calculated probability • Uniform probability • Equally likely outcomes • Random selection

**15-3** Impossibility • Certainty • Subscript

**15-5** Mutually exclusive events

**15-7** Compound event • Tree diagram • List of ordered pairs • Graph of ordered pairs • Counting Principle • Independent events

**15-8** Without replacement • Dependent events • With replacement • Conditional probability

**15-9** Permutation • Factorial symbol (!) • $n$ factorial

**15-11** Combination

## REVIEW EXERCISES

**1.** In a dish of jellybeans, some are black. Aaron takes a jellybean from the dish at random without looking at what color he is taking. Jake chooses the same color jellybean that Aaron takes. Is the probability that Aaron takes a black jellybean the same as the probability that Jake takes a black jellybean? Explain your answer.

**2.** The probability that Aaron takes a black jellybean from the dish is $\frac{8}{25}$. Can we conclude that there are 25 jellybeans in the dish and 8 of them are black? Explain why or why not.

**3.** The numerical value of ${}_nP_r$ is the product of $r$ factors. Express the smallest of those factors in terms of $n$ and $r$.

**4.** If $P(A) = .4$, $P(B) = .3$, and $P(A \cap B) = .2$, find $P(A \cup B)$.

**5.** If $P(A) = .5$, $P(B) = .2$, and $P(A \cup B) = .5$, find $P(A \cap B)$.

In 6–11, evaluate each expression:

**6.** 8! **7.** ${}_5P_5$ **8.** ${}_{12}P_2$ **9.** ${}_5C_5$ **10.** ${}_{12}C_3$ **11.** ${}_{40}C_{38}$

**12.** How many 7-letter words can be formed from the letters in UNUSUAL if each letter is used only once in a word?

**13.** A SYZYGY is a nearly straight-line configuration of three celestial bodies such as the Earth, Moon, and Sun during an eclipse.

**a.** How many different 6-letter arrangements can be made using the letters in the word SYZYGY?

**b.** Find the probability that the first letter in an arrangement of SYZYGY is

(1) G (2) Y (3) a vowel (4) not a vowel

**14.** From a list of 10 books, Gwen selects 4 to read over the summer. In how many ways can Gwen make a selection of 4 books?

**15.** Mrs. Moskowitz, the librarian, checks out the 4 books Gwen has chosen. In how many different orders can the librarian stamp these 4 books?

**16.** A coach lists all possible teams of 5 that could be chosen from 8 candidates. How many different teams can he list?

**17.** The probability that Greg will get a hit the next time at bat is 35%. What is the probability that Greg will *not* get a hit?

**18. a.** How many 3-digit numerals can be formed using the digits 2, 3, 4, 5, 6, 7, and 8 if repetition is not allowed?

**b.** What is the probability that such a 3-digit numeral is greater than 400?

**19. a.** If a 4-member committee is formed from 3 girls and 6 boys in a club, how many committees can be formed?

**b.** If the members of the committee are chosen at random, find the probability that the committee consists of:

(1) 2 girls and 2 boys (2) 1 girl and 3 boys (3) 4 boys

**c.** What is the probability that the fourth member chosen is a girl if the first three are 2 boys and a girl?

**d.** What is the probability that a boy is on the committee?

**20.** From a 52-card deck, 2 cards are drawn at random without replacement. Find the probability of selecting:

**a.** a ten and a king, in that order
**b.** 2 tens
**c.** a ten and a king, in either order
**d.** 2 spades
**e.** a king as the second card if the king of hearts is the first card selected.

**21.** If a card is drawn at random from a 52-card deck, find the probability that the card is:

**a.** an eight or a queen
**b.** an eight or a club
**c.** red or an eight
**d.** not a club
**e.** red and a club
**f.** not an eight or not a club

**22.** From an urn, 3 marbles are drawn at random with no replacement. Find the probability that the 3 marbles are the same color if the urn contains, in each case, the given marbles:

**a.** 3 red and 2 white
**b.** 2 red and 2 blue
**c.** 3 red and 4 blue
**d.** 4 white and 5 blue
**e.** 10 blue
**f.** 2 red, 1 white, 3 blue

**23.** From a class of girls and boys, the probability that one student chosen at random to answer a question will be a girl is $\frac{1}{3}$. If four boys leave the class, the probability that a student chosen at random to answer a question will be a girl is $\frac{2}{5}$. How many boys and girls are there in the class before the four boys leave?

**24.** A committee of 5 is to be chosen from 4 men and 3 women.

**a.** How many different 5-person committees are possible?
**b.** Find the probability that the committee includes:
(1) 2 men and 3 women
(2) 3 men and 2 women
(3) at least 2 women
(4) all women
**c.** In how many ways can this 5-person committee select a chairperson and a secretary?
**d.** If Hilary and Helene are on the committee, what is the probability that one is selected as chairperson and the other as secretary?

**25.** Assume that $P(\text{male}) = P(\text{female})$. In a family of three children, what is the probability that all three children are of the same gender?

**26.** **a.** Let $n(S)$ = the number of five-letter words that can be formed using the letters in the word RADIO. Find $n(S)$.
**b.** Let $n(E)$ = the number of five-letter words that can be formed using the letters in the word RADIO, in which the first letter is a vowel Find $n(E)$.

**c.** If the letters in RADIO are rearranged at random, find the probability that the first letter is a vowel by using the answers to parts **a** and **b**,

**d.** Find the answer to part **c** by a more direct method.

In 27–32, if the letters in each given word are rearranged at random, use two different methods to find the probability that the first letter in the word is A.

**27.** RADAR **28.** CANVAS **29.** AZALEA

**30.** AA **31.** DEFINE **32.** CANOE

**33.** Twenty-four women and eighteen men are standing in a ticket line. What is the probability that the first five persons in line are women? (The answer need not be simplified.)

**34.** A four-digit code consists of numbers selected from the set of even digits: 0, 2, 4, 6, 8. No digit is used more than once in any code and the code can begin with 0. What is the probability that the code is a number less than 4,000?

**35.** In a class there are 4 more boys than girls. A student from the class is chosen at random. The probability that the student is a boy is $\frac{3}{5}$. How many girls and how many boys are there in the class?

**36.** The number of seniors in the chess club is 2 less than twice the number of juniors, and the number of sophomores is 7 more than the number of juniors. If one person is selected at random to represent the chess club at a tournament, the probability that a senior is chosen is $\frac{2}{5}$. Find the number of students from each class who are members of the club.

**37.** In a dish, Annie has 16 plain chocolates and 34 candy-coated chocolates, of which 4 are blue, 12 are purple, 15 are green, and 3 are gray.

**a.** What is the probability that Annie will randomly choose a plain chocolate followed by a blue chocolate?

**b.** Annie eats 2 green chocolates and then passes the dish to Bob. What is the probability that he will randomly choose a purple chocolate?

**38.** Find the probability that the next patient of the doctor described in the chapter opener on page 575 will need either a flu shot or a pneumonia shot.

## Exploration

Rachael had a box of disks, all the same size and shape. She removed 20 disks from the box, marked them, and then returned them to the box. After mixing the marked disks with the others in the box, she removed a handful of disks and recorded the total number of disks and the number of marked disks. Then she returned the disks to the box, mixed them, and removed another handful. She repeated this last step eight times. The chart below shows her results.

| Disks | 12` | 10 | 15 | 11 | 8 | 10 | 7 | 12 | 9 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|
| Marked Disks | 3 | 4 | 5 | 4 | 3 | 2 | 2 | 5 | 3 | 5 |

**a.** Use the data to estimate the number of disks in the box.

**b.** Repeat Rachael's experiment using a box containing an unknown number of disks. Compare your estimate with the actual number of disks in the box.

**c.** Explain how this procedure could be used to estimate the number of fish in a pond.

## CUMULATIVE REVIEW — CHAPTERS 1–15

### Part I

Answer all questions in this part. Each correct answer will receive 2 credits. No partial credit will be allowed.

**1.** Which of the following is a rational number?

(1) $\sqrt{169}$ (2) $\sqrt{12}$ (3) $\sqrt{9+4}$ (4) $\pi$

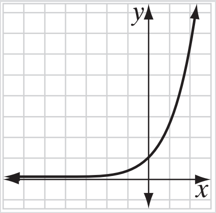
**2.** The area of a trapezoid is 21 square inches and the measure of its height is 6.0 inches. The sum of the lengths of the bases is

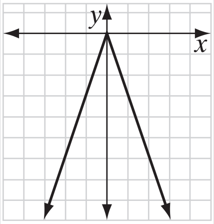
(1) 3.5 inches (2) 5.3 inches (3) 7.0 inches (4) 14 inches

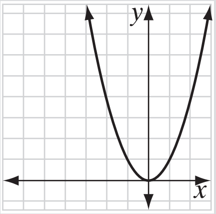
**3.** A point on the line whose equation is $y = -3x + 1$ is

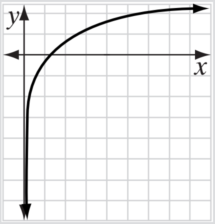
(1) $(4, -1)$ (2) $(-1, -2)$ (3) $(-2, 1)$ (4) $(1, -2)$

**4.** Which of the following is the graph of an exponential function?

(1)



(3)



(2)



(4)


**5.** The height of a cone is 12.0 centimeters and the radius of its base is 2.30 centimeters. What is the volume of the cone to three significant digits?

(1) 52.2 square centimeters
(2) 52.3 square centimeters
(3) 66.5 square centimeters
(4) 66.6 square centimeters

**6.** If $12 - 3(x - 1) = 7x$, the value of $x$ is

(1) 1.5 (2) 4.5 (3) 0.9 (4) 3.75

**7.** The product of $4x^{-2}$ and $5x^3$ is

(1) $20x^{-6}$ (2) $20x$ (3) $9x^{-6}$ (4) $9x$

**8.** There are 12 members of the basketball team. At each game, the coach selects a group of five team members to start the game. For how many games could the coach make different selections?

(1) 12! (2) 5! (3) $\frac{12!}{5!}$ (4) $\frac{12!}{7! \cdot 5!}$

**9.** Which of the following is an example of direct variation?

(1) the area of any square compared to the length of its side
(2) the perimeter of any square compared to the length of its side
(3) the time it takes to drive 500 miles compared to the speed
(4) the temperature compared to the time of day

**10.** Which of the following intervals represents the solution set of the inequality $-3 \leq 2x+1 < 7$?

(1) $(-3, 7)$ (2) $(-2, 3)$ (3) $[-3, 7)$ (4) $[-2, 3)$

## Part II

Answer all questions in this part. Each correct answer will receive 2 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

**11.** The vertices of parallelogram $ABCD$ are $A(0, 0)$, $B(7, 0)$, $(12, 8)$, and $D(5, 8)$. Find, to the nearest degree the measure of $\angle DAB$.

**12.** Marty bought 5 pounds of apples for 98 cents a pound. A week later, she bought 7 pounds of apples for 74 cents a pound. What was the average price per pound that Marty paid for apples?

## Part III

Answer all questions in this part. Each correct answer will receive 3 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

**13.** Each student in an international study group speaks English, Japanese, or Spanish. Of these students, 100 speak English, 50 speak Japanese, 100 speak Spanish, 45 speak English and Spanish, 10 speak English and

Japanese, 13 speak Spanish and Japanese, and 5 speak all three languages. How many students do not speak English?

**14.** Abe, Brian, and Carmela share the responsibility of caring for the family pets. During a seven-day week, Abe and Brian each take three days and Carmela the other one. In how many different orders can the days of a week be assigned?

## Part IV

Answer all questions in this part. Each correct answer will receive 4 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

**15.** Mrs. Martinez is buying a sweater that is on sale for 25% off of the original price. She has a coupon that gives her an additional 20% off of the sale price. Her final purchase price is what percent of the original price?

**16.** The area of a rectangular parcel of land is 720 square meters. The length of the land is 4 meters less than twice the width.

**a.** Write an equation that can be used to find the dimensions of the land.

**b.** Solve the equation written in part **a** to find the dimensions of the land.